

# UNIT 4

**M**any situations involve paths and networks, like bus routes and computer networks. *Vertex-edge graphs* can be used as mathematical models to help analyze such situations. A vertex-edge graph is a diagram consisting of points (vertices) and arcs or line segments (edges) connecting some of the points. Such graphs are part of geometry, as well as part of an important contemporary field called *discrete mathematics*.

In this unit, you will use vertex-edge graphs and Euler circuits to help find optimum paths, such as the best route to collect money from parking meters, deliver newspapers, or plow snow from city streets. You will also use vertex coloring of graphs to avoid conflict among objects, such as scheduling conflicts among meetings or broadcast interference among nearby radio stations.

You will develop the understanding and skill needed to solve problems about optimum paths and conflict through your work in two lessons.

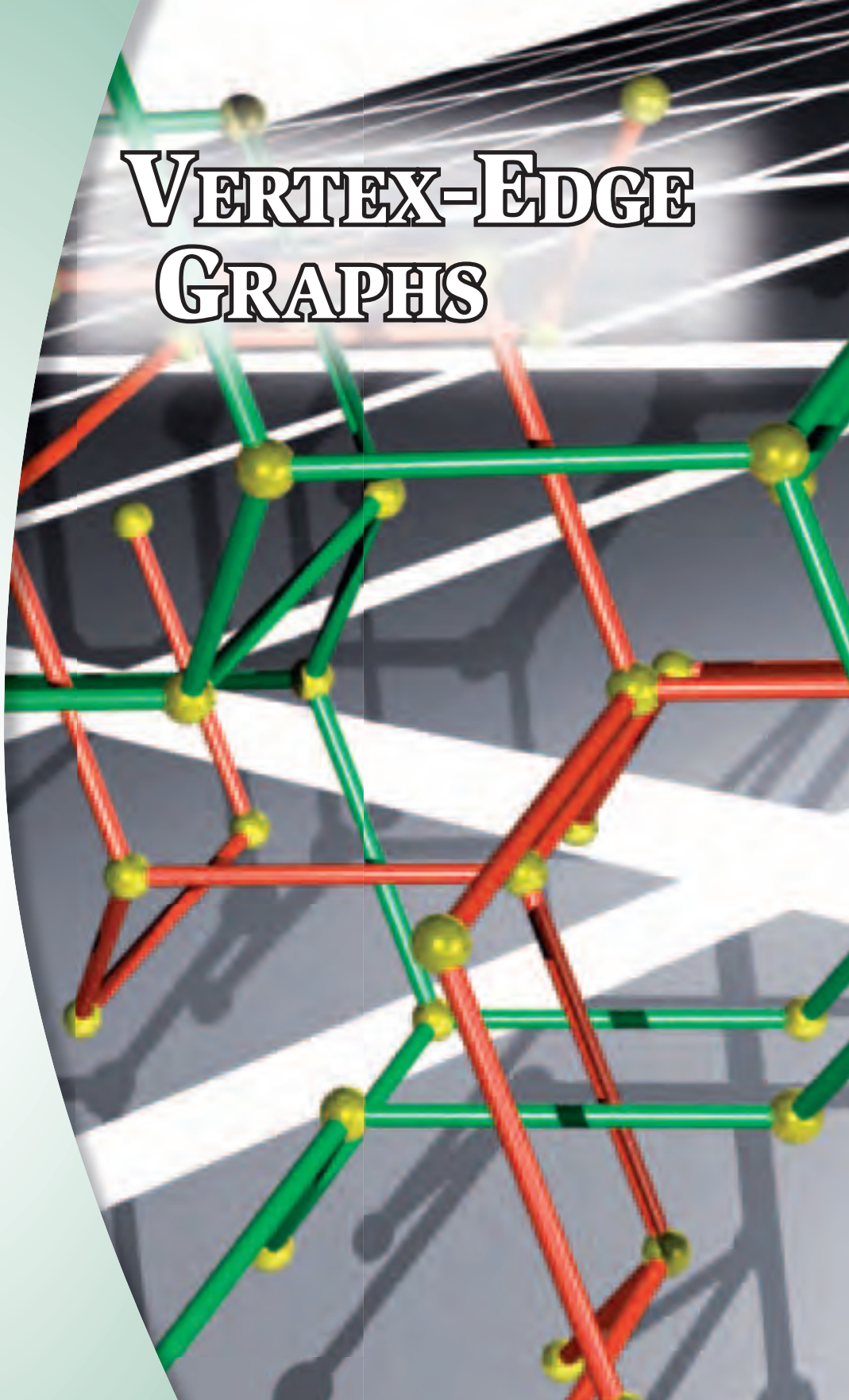
## Lessons

### 1 *Euler Circuits: Finding the Best Path*

Use Euler circuits and their properties to solve problems about optimum circuits.

### 2 *Vertex Coloring: Avoiding Conflict*

Use vertex coloring to solve problems related to avoiding conflict in a variety of settings.



# LESSON 1

## *Euler Circuits: Finding the Best Path*



Source: London Transport Museum Guide

Often when solving problems using geometric figures or diagrams, you are concerned with their size, shape, or position. However, sometimes a geometric diagram is used to represent a situation in which size, shape, and position are not important. Instead, connections are what really matter. In this unit, you will study geometric diagrams made up of vertices and edges in which size and shape are not essential characteristics, but how the vertices are connected is very important.

A subway map is one common example of a geometric diagram with vertices and edges for which precise size and shape are not crucial. Perhaps the first such map was the 1933 London Underground map shown above.

## Think About This Situation

Examine the Underground map and think about how a visitor to London might have used the map.

- What information is conveyed by the map? What information about the city is not conveyed by the map?
- Why are the size and shape of the map layout not essential?
- What are important features of subway maps like the one above?
- Describe other geometric diagrams with vertices and edges for which *connections* are important, but exact size and shape are not essential.

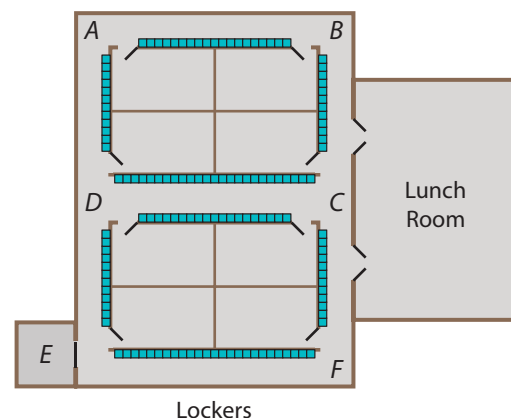
Geometric diagrams made up of vertices and edges, in which connections are important but exact size, shape, and position are not essential, are sometimes called *vertex-edge graphs*, or simply *graphs*. In this lesson, you will learn how to use vertex-edge graphs to find optimum routes.

### Investigation 1 Planning Efficient Routes

You can save time, energy, and expense by studying a complex project before you begin your work. There may be many ways to carry out the project. However, one way may be judged to be the “best” or *optimum*, in some sense. As you work on this investigation, think about this question:

*How can you create and use a mathematical model to find an optimum solution to problems such as the following locker-painting problem?*

**Locker Painting** Suppose you are hired to paint all the lockers around eight classrooms on the first floor of a high school. The lockers are located along the walls of the halls as shown in the diagram to the right. Letters are placed at points where you would stop painting one row of lockers and start painting another. Five-gallon buckets of paint, a spray paint compressor, and other equipment are located in the first-floor equipment room *E*. You must move this bulky equipment with you as you paint the lockers. You also must return it to the equipment room when you are finished painting. (The lockers in the center hall must be painted one side at a time.)





- 1 Since you are being paid for the job, not by the hour, you would like to paint the lockers as quickly and efficiently as possible.
  - a. Which row would you paint first? Is there more than one choice for the first row to paint?
  - b. Which row would you paint last? Why?

- 2 Here are three plans that have been suggested for painting the lockers.

**Plan I:** Paint from  $E$  to  $F$ ,  $F$  to  $C$ ,  $C$  to  $D$  (one side),  $D$  to  $E$ ,  $D$  to  $A$ ,  $A$  to  $B$ ,  $B$  to  $C$ ,  $C$  to  $D$  (the other side).

**Plan II:** Paint from  $A$  to  $B$ ,  $B$  to  $C$ ,  $C$  to  $D$  (one side),  $D$  to  $A$ ,  $D$  to  $C$  (other side),  $C$  to  $F$ ,  $F$  to  $E$ ,  $E$  to  $D$ .

**Plan III:** Paint from  $E$  to  $D$ ,  $D$  to  $A$ ,  $A$  to  $B$ ,  $B$  to  $C$ ,  $C$  to  $F$ ,  $F$  to  $E$ ,  $D$  to  $C$  (one side),  $C$  to  $D$  (other side).

- a. Which, if any, of these plans do you think is optimum; that is, the “best” way to do the painting? If a plan is not optimum, explain why not.
- b. Without help from your classmates, prepare a plan you think is optimum for painting the lockers.
- c. Compare your plan with those of others.
  - i. How are they alike? How are they different?
  - ii. Agree on a list of criteria that can be used to decide whether a plan is optimum.

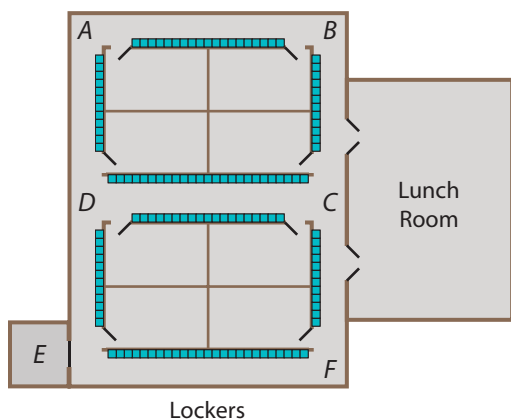
- 3 A **mathematical model** is a symbolic or pictorial representation including only the essential features of a problem situation. The floor-plan map of the first floor of the school shows the rows of lockers, classrooms, lunch room, equipment room, hallways, and outer walls. There are some features of this map that you do not need in order to solve the locker-painting problem.

- a. Which of the features of the map did you use as you tried to solve the locker-painting problem? Which features were not needed?
- b. Refer to the first-floor map of the school above. Think about a simplified diagram (a mathematical model) that includes only the essential features of the locker-painting problem. For example, the lettered points on the map are important because  $E$  is the beginning and ending point, and the other letters mark where one row of lockers ends and another begins. Complete a copy of the diagram below so that it is a mathematical model for the locker-painting problem.

$A \bullet \qquad \bullet B$

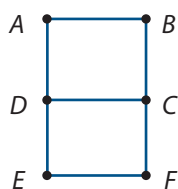
$D \bullet \qquad \bullet C$

$E \bullet \qquad \bullet F$

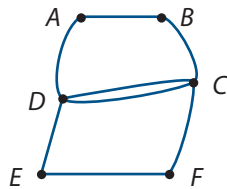


A diagram consisting of a set of points along with segments or arcs joining some of the points is called a **vertex-edge graph**, or simply a **graph**. The points are called **vertices**, and each point is called a **vertex**. The segments or arcs joining the vertices are called **edges**. The word “graph” is used to mean different things at different times in mathematics. In this unit, the word “graph” typically refers to a diagram consisting of vertices and edges.

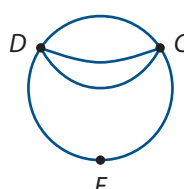
- 4** Now examine the vertex-edge graph models drawn by some other students.



**Michael's Model**



**Deonna's Model**



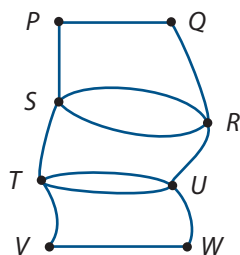
**Amy's Model**

- Does Michael's vertex-edge graph show all the essential features of the locker-painting problem? If so, explain. If not, describe what is needed.
- Is Deonna's vertex-edge graph an appropriate model for the locker-painting problem? Why do you think Deonna joined vertices C and D with 2 edges?
- Is Amy's graph an appropriate model for the locker-painting problem? Explain.

- 5** In Problem 2, you were asked to find an optimum plan for painting the first-floor lockers.

- Use that plan to trace an optimum painting route on the vertex-edge graph you drew in Problem 3. If you cannot trace your optimum route on your graph, carefully check both your optimum plan and your graph.
- Trace the same painting route on Deonna's graph. Does it matter if the vertices are connected by straight line segments or curved arcs? Does it matter how long the edges are?

- 6** Below is a vertex-edge graph that models a different arrangement of lockers.



- Draw a school floor-plan map that corresponds to this graph. Assume that the equipment room is at V.
- Find, if possible, an optimum route for painting these lockers.

## Summarize the Mathematics

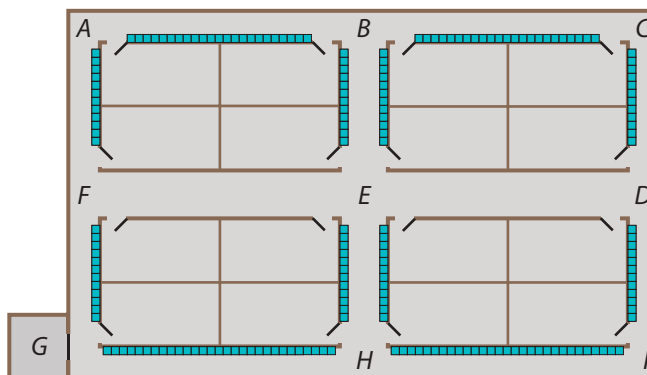
In this investigation, you explored how vertex-edge graphs can be used to model situations in which an efficient route is to be found.

- What is the difference between a floor-plan map of a school showing the lockers to be painted and a mathematical model of the locker-painting problem?
- A key step in modeling a problem situation with a graph is to decide what the vertices and edges will represent. Refer back to Problem 4. Both Deonna's and Amy's models are appropriate. What do the vertices and edges represent in each of these graphs in terms of the locker-painting problem?
- Can two vertex-edge graphs that have different shapes represent the same problem situation? Can two graphs that have different numbers of vertices and edges represent the same problem situation? Explain.
- In Problem 2, you wrote a list of criteria for an optimum locker-painting plan. Restate those criteria in terms of tracing around a vertex-edge graph that models the situation.

*Be prepared to share your ideas with the entire class.*

### ✓ Check Your Understanding

Suppose the lockers and an equipment room on the west wing of a high school are located as shown below.



- If you were to model the problem of painting these lockers with a vertex-edge graph, what would the vertices represent? The edges?
- Draw a graph that models this problem.
- Determine an optimum plan for painting the lockers. Check your plan against the criteria for tracing the edges and vertices of a graph that you prepared in Part d of the Summarize the Mathematics.

## Investigation 2 Making the Circuit

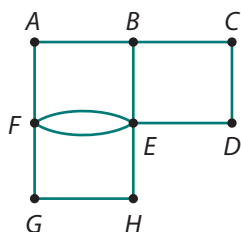
Your criteria for the optimum sequence for painting the lockers in the previous investigation are the defining characteristics of an important property of a graph. An **Euler** (pronounced oy' lur) **circuit** is a route through a connected graph such that (1) each edge of the graph is traced exactly once, and (2) the route starts and ends at the same vertex.

You only consider Euler circuits in connected graphs. A **connected graph** is a graph that is all in one piece. That is, from each vertex there is at least one path to every other vertex. Given a connected graph, it often is helpful to know if it has an Euler circuit. (The name "Euler" is in recognition of the eighteenth-century Swiss mathematician Leonhard Euler. He was the first to study and write about these circuits.) As you work through this investigation, look for clues that help you answer these questions:

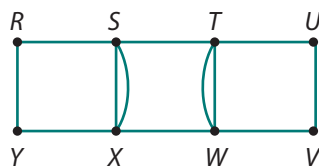
*How can you tell if a graph has an Euler circuit?*

*If a graph has an Euler circuit, how can you systematically find it?*

- 1 Graph models of the sidewalks in two sections of a town are shown below. Parking meters are placed along these sidewalks.



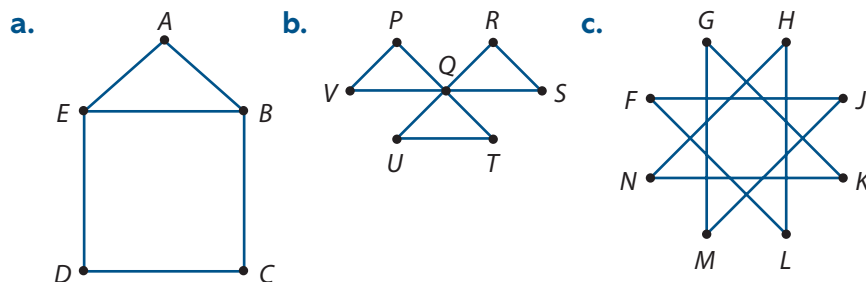
East Town Model



West Town Model

- a. Why would it be helpful for a parking-control officer to know if these graphs have Euler circuits?
  - b. Does the graph that models the east section of town have an Euler circuit? Explain your reasoning.
  - c. Does the graph that models the west section of town have an Euler circuit? Does it have more than one Euler circuit? Explain.
- 2 The three graphs at the top of the next page are similar to puzzles enjoyed by people all over the world. In each case, the challenge is to trace the figure. You must trace every edge exactly once without lifting your pencil and return to where you started. That is, the challenge is to trace an Euler circuit through the figure or graph. Place a sheet of paper over each graph and try to trace an Euler circuit. If the graph has an Euler circuit, write down the vertices in order as you trace the circuit. (Note that in the graph in Part c, only the points with letter labels are vertices. The other edge crossings are not vertices; you can think of them like overpasses in a road system.)





**3** By looking at the form of a function rule, you often can predict the shape of the graph of the function without plotting any points. Similarly, it would be helpful to be able to examine a vertex-edge graph and predict if it has an Euler circuit without trying to trace it.

- Have each member of your group draw a graph with five or more edges that has an Euler circuit. On a separate sheet of paper, have each group member draw a connected graph with five or more edges that does *not* have an Euler circuit. Alternatively, use vertex-edge graph software to generate several graphs that have an Euler circuit and several graphs that do not.
- Sort your group's graphs into two collections, those that have an Euler circuit and those that do not.
- Examine the graphs in the two collections. Describe key ways in which graphs that have Euler circuits differ from those that do not.
- Try to figure out a way to predict if a graph has an Euler circuit simply by examining its vertices.
  - Test your method of prediction using the graphs in Problem 2.
  - If you have access to vertex-edge graph software, generate several additional general graphs to test your method.



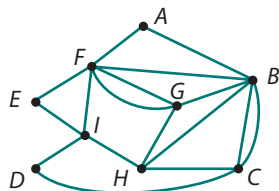
- Make a conjecture about the properties of a graph that has an Euler circuit. Explain why you think your conjecture is true for *any* graph that has an Euler circuit. Compare and test your conjecture with other students' conjectures and graphs. Modify your conjecture and explanation as needed.



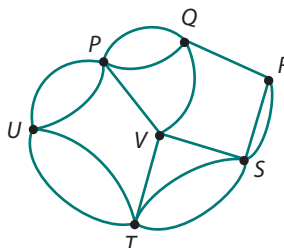
- 4 In your conjecture from Problem 3 Part e about which graphs have an Euler circuit, you probably counted the number of edges at each vertex of the graph. The number of edges touching a vertex is called the **degree of the vertex**. Restate your conjecture in terms of the degrees of the vertices. (If an edge loops back to the same vertex, that counts as two edge touchings. For an example see Extensions Task 29 on page 262.)

- 5 Once you can predict whether a graph has an Euler circuit, it is often still necessary to find the circuit. Consider the graphs below.

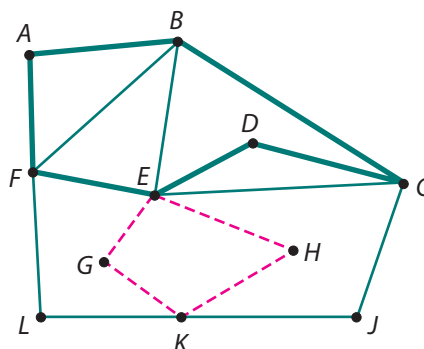
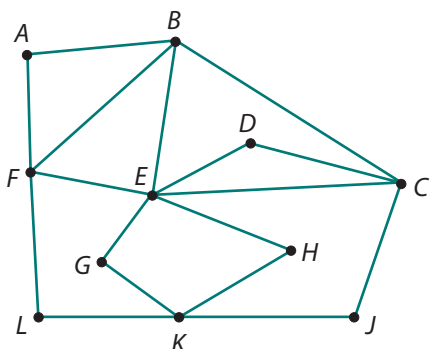
**Graph I**



**Graph II**



- For each graph, predict whether it has an Euler circuit.
  - If the graph has an Euler circuit, find it.
  - Describe the method you used to find your Euler circuit. Describe other possible methods for finding Euler circuits.
- 6 One systematic method for finding an Euler circuit is to trace the circuit in stages. For example, suppose you and your classmates want to find an Euler circuit that begins and ends at A in the graph below. You can trace the circuit in several stages.



**Stage I:** Alicia began by drawing a circuit that begins and ends at A. The circuit she drew, shown in the diagram by the heavy edges, was  $A-B-C-D-E-F-A$ . But this does not trace all edges.

**Stage II:** George added another circuit shown by the dashed edges starting at E:  $E-G-H-E$ .

- Alicia's and George's circuits can be combined to form a single circuit beginning and ending at A. List the order of vertices for that combined circuit.

**Stage III:** Since this circuit still does not trace each edge, a third stage is required.

- b.** Trace a third circuit which covers the rest of the edges.
- c.** Combine all the circuits to form an Euler circuit that begins and ends at A. List the vertices of your Euler circuit in order.
- d.** Use this method to find an Euler circuit in Graph I of Problem 5.

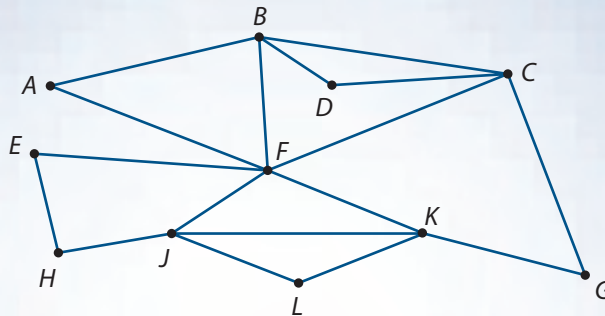
- 7** Choose your preferred method for finding Euler circuits from Problems 5 and 6. Write specific step-by-step instructions that describe the method you chose. Your instructions should be written so that they apply to *any* graph, not just the one that you may be working on at the moment. Such a list of step-by-step instructions is called an **algorithm**.

Creating algorithms is an important aspect of mathematics. Algorithms are especially important when programming computers to solve problems. Two questions you should ask about any algorithm are *Does it always work?* and *Is it efficient?* You will consider these questions in more detail in the next lesson.

## Summarize the Mathematics

It is possible to examine a graph to decide if it has an Euler circuit. If it does, there are algorithms to find such a circuit.

- a** How can you tell if a graph like the one below has an Euler circuit without actually trying to trace the graph?

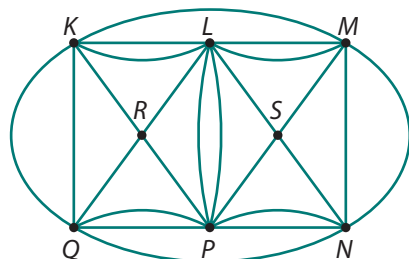


- b** Use your algorithm from Problem 7 to find an Euler circuit in the graph shown here.  
**Be prepared to explain your thinking and algorithm to the class.**

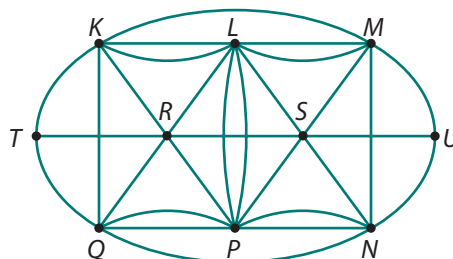
## ✓ Check Your Understanding

For each of the graphs below, decide if the graph has an Euler circuit. If there is an Euler circuit, use your algorithm to find it. If not, explain how you know that no Euler circuit exists.

a.



b.



## Investigation 3 Graphs and Matrices

Information is often organized and displayed in tables. The use of tables to summarize information can be seen in almost every section of most newspapers. As you work on the problems in this investigation, look for answers to this question:

*How can table-like arrays be used to represent vertex-edge graphs and help reason about information contained in the graphs?*

- 1 Examine this information on gold (1st place), silver (2nd place), and bronze (3rd place) medals awarded in the 2006 Winter Olympics.

**Medal Count**

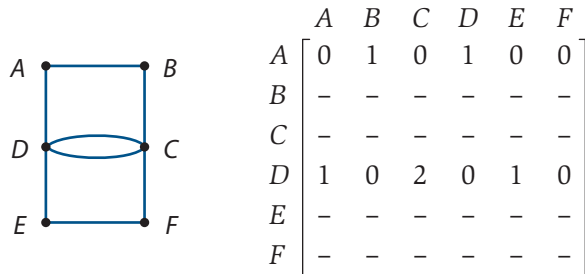
Country	G	S	B	Total
Germany	11	12	6	29
United States	9	9	7	25
Austria	9	7	7	23
Russian Fed.	8	6	8	22
Canada	7	10	7	24
Sweden	7	2	5	14

Source: [www.torino2006.org](http://www.torino2006.org)

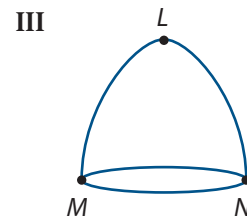
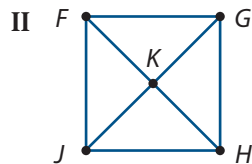
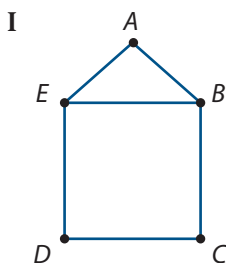


- a. What do each of the numbers in the row labeled Germany represent?
- b. What is the meaning of the number in the fifth row and second column? (Don't count the row and column headings.) In the third row and third column?
- c. Finland did not win any gold medals. However, the Finnish team did take home 6 silver medals and 3 bronze medals. How could you modify this chart to include this additional information?

- 2 Rectangular arrays of numbers, like the one below, are sometimes called **matrices**. Matrices can be used to represent graphs. One way in which a graph can be represented by a **matrix** is shown by the partially completed matrix below.



- Study the first and fourth rows of the matrix. Explain what each entry means in terms of the graph.
- Copy the matrix and then fill in the missing entries.
- Vertices in a graph that are connected by an edge are said to be **adjacent vertices**. The matrix you constructed in Part b is called an **adjacency matrix** for the graph, since it contains information about vertices that are adjacent. Each entry in an adjacency matrix is the number of direct connections (edges) between the corresponding pair of vertices. Construct an adjacency matrix for each of the three graphs below.



- It is time-consuming to construct an adjacency matrix for a large graph. If you have access to vertex-edge graph software, construct an adjacency matrix for several graphs (without loops) that contain at least 6 vertices. Click on the entries of the matrix to see the corresponding edge(s) in the graph. Construct a circuit in the graph (not necessarily an Euler circuit) by clicking on entries in the adjacency matrix.

- 3 Now examine some common properties of the adjacency matrices you have constructed.

- The **main diagonal** of a matrix like these consists of the entries in the diagonal running from the top-left corner of the matrix to the bottom-right corner. What do you notice about the main diagonal in these adjacency matrices? Explain this pattern. (The graphs you have worked with so far do not contain loops. To see how loops affect an adjacency matrix, see Extensions Task 28 on page 262.)
- Describe and explain any symmetry you see in these adjacency matrices.



- 4 The sums of the numbers in each row of a matrix are called the **row sums** of the matrix.
- Find the row sums of each of the adjacency matrices in Problem 2 Part c. What do these row sums represent in the graphs?
  - Is it possible to tell by looking at the adjacency matrix for a graph whether the graph has an Euler circuit? Justify your response.

## Summarize the Mathematics

In this investigation, you learned how a matrix can be used to represent and help analyze a graph.

- An adjacency matrix corresponding to a graph that has 5 vertices,  $A$ ,  $B$ ,  $C$ ,  $D$ , and  $E$ , listed in the matrix in that order, has a 2 in the third row, fifth column. What does the 2 represent? What does a 1 in the first row, second column mean?
- How do the row sums of adjacency matrices for graphs that do and do not have Euler circuits differ? Explain.

*Be prepared to share your thinking with the entire class.*

## ✓ Check Your Understanding

Examine the adjacency matrices below, and answer the following questions.

$$\text{I} \quad \begin{array}{c|cc} & A & B & C \\ \hline A & 0 & 2 & 0 \\ B & 2 & 0 & 1 \\ C & 0 & 1 & 0 \end{array}$$

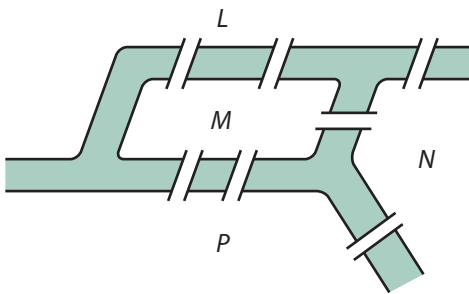
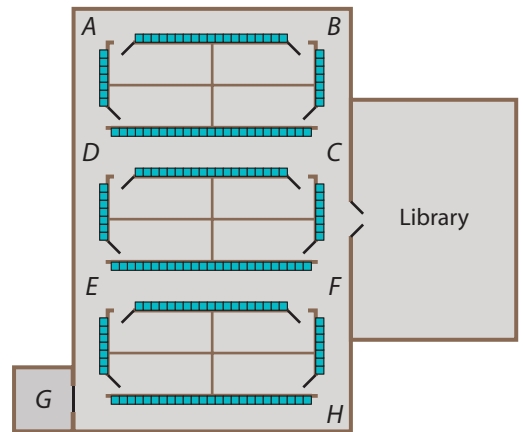
$$\text{II} \quad \begin{array}{c|cccc} & P & Q & R & S \\ \hline P & 0 & 1 & 2 & 1 \\ Q & 1 & 0 & 1 & 2 \\ R & 2 & 1 & 0 & 2 \\ S & 1 & 2 & 2 & 0 \end{array}$$

- Does each of the graphs with an adjacency matrix given above have an Euler circuit? How can you tell without drawing the graphs?
- Draw and label a graph corresponding to each adjacency matrix. Find an Euler circuit if there is one.

# On Your Own

## Applications

- 1 Suppose the lockers on the second floor of a high school are located as shown at the right. Suppose the equipment room located at G is at the bottom of a stairway leading to the second floor. Find two optimum plans for painting the lockers that satisfy the optimum criteria you listed in Investigation 1, Problem 2 Part c (page 240).



- 2 The Pregolya River runs through the Russian city of Kaliningrad. In the eighteenth century, the river was called the Pregel and the city was named Königsberg. Four parts of the city were connected by 7 bridges as illustrated here. Citizens often took walking tours of the city by crossing over the bridges. Some people wondered whether it was possible to tour the city by beginning at a point on land, walking across each bridge exactly once, and returning to the same point. This problem, called the *Königsberg bridges problem*, intrigued the mathematician Leonhard Euler, who lived at that time. The paper Euler wrote in 1736 containing the solution to this problem is widely considered to be the first paper on the theory of vertex-edge graphs. Ironically, Euler did not use an actual vertex-edge graph diagram as part of his solution, although his results clearly apply to such graphs. (Euler considered his work on this problem to be part of an area of mathematics informally discussed then as “the geometry of position,” since it dealt with relative position, but not distance.)

- a. Draw a graph in which the vertices represent the 4 land areas (lettered in the figure) and the edges represent bridges.
- b. What is the solution to the Königsberg bridges problem? Explain your response.
- c. In the time since Euler solved the problem, two more bridges were built. One bridge was added at the left to connect areas labeled L and P. Another bridge was added to connect areas labeled N and P.
  - i. Draw a graph that models this new situation of land areas and bridges.
  - ii. Use your graph to determine if it is possible to take a tour of the city that crosses each of the 9 bridges exactly once and allows you to return to the point where you started.

- 3** The Bushoong are a subgroup of the Kuba chiefdom in the Democratic Republic of Congo (changed from Zaire in 1997). Bushoong children have a long tradition of playing games that involve tracing figures in the sand using a stick. The challenge is to trace each line once and only once without lifting the stick from the sand. Two such figures are given below. (Problem adapted from *Ethnomathematics: A Multicultural View of Mathematical Ideas*, Brooks/Cole Publishing Company, 1991.)



Figure I

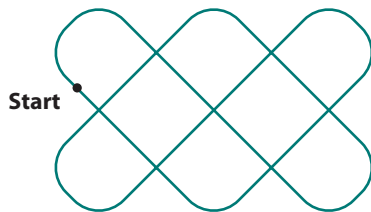
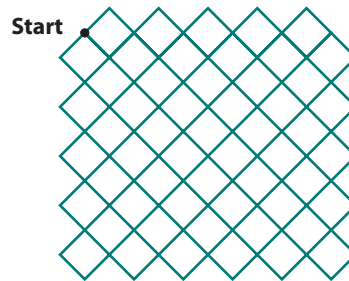
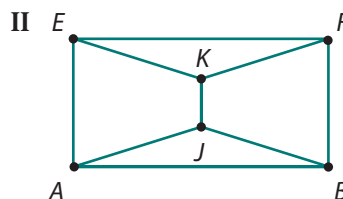
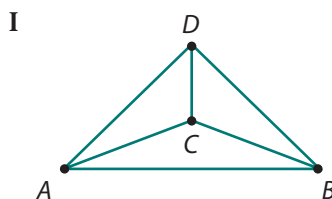


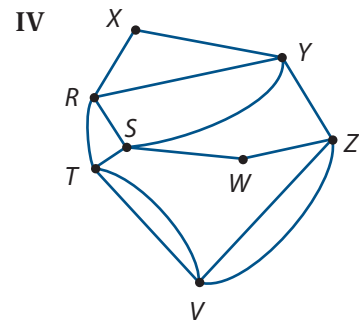
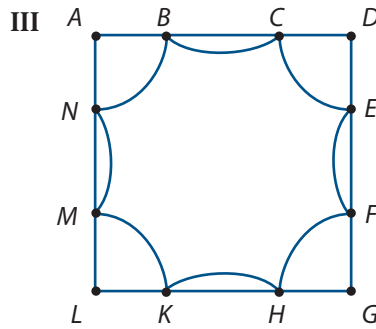
Figure II



Place a sheet of paper over the figures.

- Trace each figure without lifting your pencil and without any retracing. Your tracing does not need to end at the same place it started.
  - Try tracing each figure using different “start” points. Summarize your findings.
- 4** Some popular puzzles involve trying to trace a figure starting and ending at the same vertex without lifting your pencil or tracing an edge more than once. That is, you try to find an Euler circuit.
- Identify which of the following graphs do not have an Euler circuit. Explain why they do not.

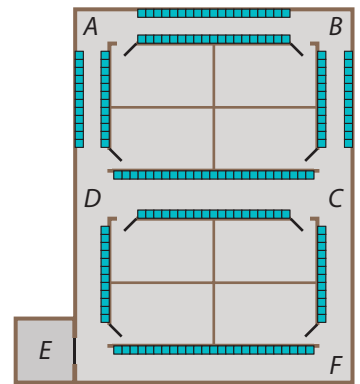




- b.** For each of the graphs that has an Euler circuit, use the algorithm you developed to find a circuit. Write down the sequence of vertices visited as you trace the circuit.
- c.** Draw two graphs that would be difficult or impossible to trace without lifting your pencil from the page or tracing an edge more than once—draw one so that it has an Euler circuit and the other so that it does not.
- d.** Use your graphs from Part c to amaze and teach someone outside of class, as follows. Challenge some people to trace your graphs, without lifting their pencil or tracing any edge more than once and starting and ending at the same point. Then ask them to challenge you in the same way with any graph they draw. See if you can amaze them with how quickly you can tell whether or not it is possible to trace the graph. Then teach them about Euler circuits so they will know the secret too.

- 5** Suppose the lockers on the second floor of the high school in the locker-painting problem on page 239 are located as shown here.

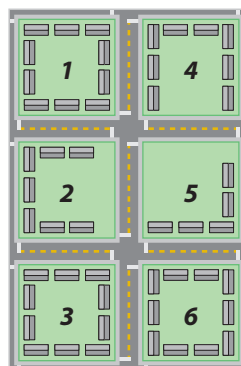
- Draw a graph that represents this situation. Be sure to describe what the vertices and edges of your graph represent.
- Is there a way to paint the lockers by starting and ending at the equipment room and never moving equipment down a hall without painting lockers on one side? Explain.
- Is there a way to paint the lockers by starting at  $D$  and ending at  $C$  and never moving equipment down a hall without painting lockers on one side? Compare the degree of  $C$  to the degrees of the other vertices. Make graphs in which there is a route through one vertex, ends at another, and traverses



### Second-Floor Lockers

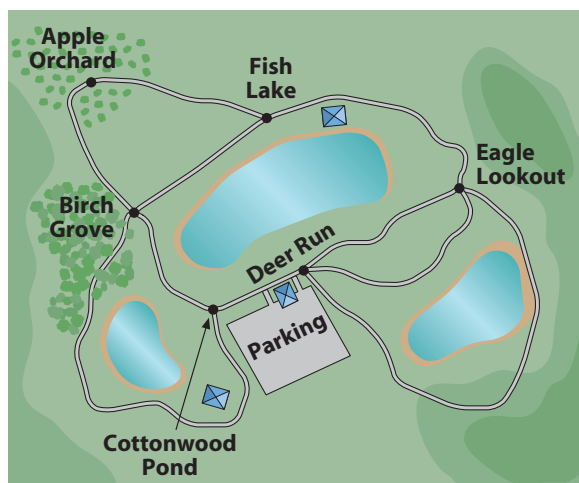
- 6** A newspaper carrier wants to complete a delivery route without retracing steps. Some streets on the route have houses facing each other. Whenever there are houses on both sides of a street, papers are delivered to both sides by making all deliveries to one side and then along the other side.





- a. Suppose the paper carrier only delivers to the houses on blocks 1, 2, and 3. Construct a vertex-edge graph model for this situation. What do the edges and vertices represent? Find an optimum delivery route.
- b. Suppose the paper carrier delivers to the houses on all 6 blocks. Construct a vertex-edge graph model for this situation. Find an optimum delivery route.
- c. Now assume that *all* blocks have houses on all 4 sides and all streets continue in both directions.
  - i. Add 3 more blocks that are adjacent to the given blocks on the street map. Find an optimum delivery route.
  - ii. Can you find an Euler circuit no matter where the 3 new blocks are placed on the route? Explain your response.
  - iii. Is it possible to place any number of new blocks on the route and still have an Euler circuit? Explain your reasoning.

- 7** The map below shows the trails in Tongas State Park. The labeled dots represent rest areas scattered throughout the park.



- How would a graph model of this situation differ from the map? Is it necessary or useful to draw a graph model in this situation? Why or why not?
- Construct an adjacency matrix related to the park map.
- Is it possible to hike each of the trails in the park once and return to your car in the parking lot? Explain your answer by using the adjacency matrix from Part b and your knowledge of Euler circuits.
- The Park Department has received money to build additional trails. Between which rest stops should they build a new trail (or trails) so that people can hike each trail once and return to their cars?

- 8** Certain towns in southern Alaska are on islands or isolated by mountain ranges. When traveling between these communities, you must take a boat or a plane. Listed below are the routes provided by a local airline.

**Routes between:**

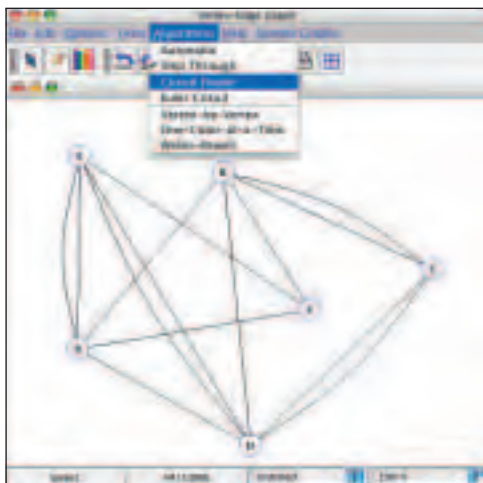
Anchorage and Cordova  
 Anchorage and Juneau  
 Cordova and Yakutat  
 Juneau and Ketchikan  
 Juneau and Petersburg  
 Juneau and Sitka  
 Petersburg and Wrangell  
 Sitka and Ketchikan  
 Wrangell and Ketchikan  
 Yakutat and Juneau



- Draw a vertex-edge graph that models this situation.
- In what ways is your graph model like the map? In what ways is it different?
- An airline inspector wants to evaluate the airline's operations by flying each route. It is sufficient to fly each route one-way. Can the inspector start in Juneau, fly all the routes exactly once, and end in Juneau?
- How would an adjacency matrix for the graph show whether or not there is a route as described in Part c?

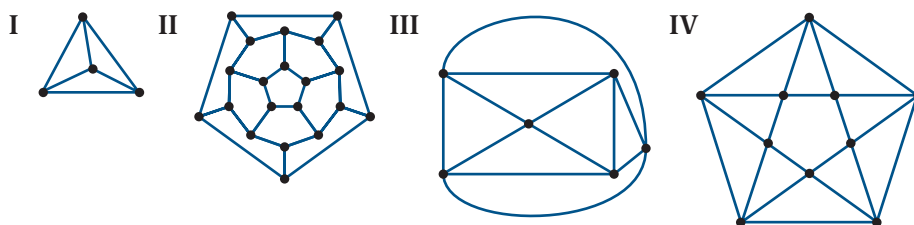
- 9** In Problem 6 of Investigation 2 (page 245), you learned about an algorithm for finding Euler circuits. Vertex-edge graph software can help you practice and understand this algorithm.

- Generate an Euler graph from the Sample Graphs menu. (Create several Duplicate Graphs so they will be available to work with as needed.) Use different colors to identify partial circuits in the graph. Then choose black to select and color the edges of the final combined circuit.
- Generate another Euler graph and use the Circuit Finder algorithm to find an Euler circuit. Use the algorithm in both Automatic and Step Through modes.



# Connections

- 10** The following graphs separate the plane into several *regions*. The exterior of the graph is an infinite region. The interior regions are enclosed by the edges. For example, Graph I separates the plane into four regions (three are enclosed by the graph and the fourth is outside the graph).



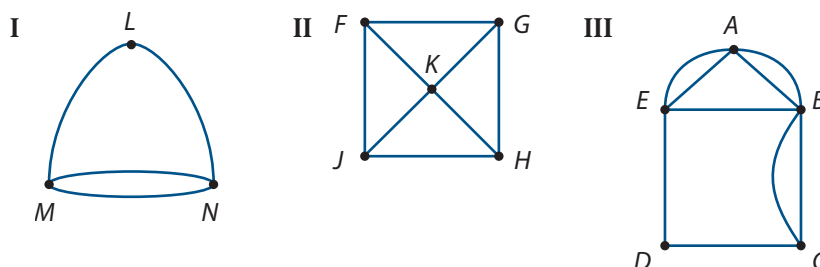
- a.** Complete a table like the one that follows using the graphs above. Be sure to count the exterior of the graph as one region.

Graph	Number of Vertices ( $V$ )	Number of Regions ( $R$ )	Number of Edges ( $E$ )
I			
II			
III			
IV			

- b.** Find a formula relating the numbers of vertices  $V$ , regions  $R$ , and edges  $E$ , by using addition or subtraction to combine  $V$ ,  $R$ , and  $E$ .
- c.** Draw several more graphs, and count  $V$ ,  $R$ , and  $E$ . Does your formula also work for these graphs?
- d.** Use your formula to predict how many regions would be formed by an appropriate graph with 5 vertices and 12 edges. Draw such a graph to verify your answer.

- 11** In this lesson, you discovered that some graphs do not have an Euler circuit.

- a.** If you are *not* required to start and end at the same vertex, do you think every graph has a path that traces every edge of the graph exactly once? Why or why not?
- b.** Place a sheet of paper over the graphs below. Try to copy the graphs by tracing each edge exactly once. You don't have to start and end at the same vertex.



- c. For those graphs that can be traced in this manner, how do the starting and ending vertices differ from the other vertices?
- d. An **Euler path** is a route through a connected graph that traces each edge of the graph exactly once. Thus an Euler circuit is a special type of Euler path, one in which the starting and ending vertices must be the same. State a rule for determining whether or not a graph has an Euler path in which the starting vertex is different from the ending vertex.

- 12** Tracing continuous figures is exhibited in cultures around the world. The Malekula live on an island in the South Pacific chain of some eighty islands that comprise the Republic of Vanuatu. As with the Bushoong in Africa (see Applications Task 3), the Malekula also have figures that represent objects or symbols of the culture. For example, Figure I below represents a yam. Figure II is called “the stone of Ambat.” (Problem adapted from *Ethnomathematics: A Multicultural View of Mathematical Ideas*, Brooks/Cole Publishing Company, 1991.)

Figure I

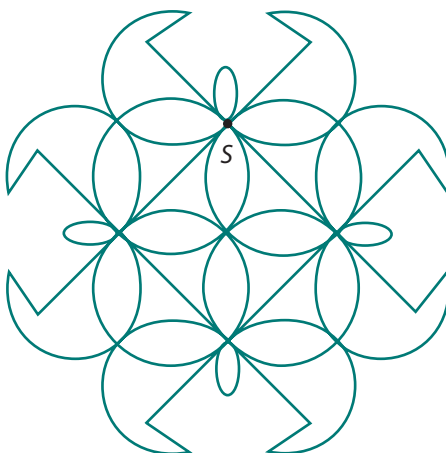
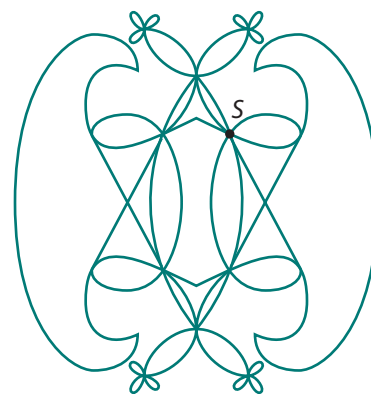


Figure II



- a. Can you trace each of these figures without lifting your pencil or tracing any edges more than once?
  - b. Describe any *symmetry* you see in each figure.
- 13** For vertex-edge graphs, the position of the vertices and the length and straightness of the edges are not critical. What is important is the way in which the edges connect the vertices. Consider the following matrix. Each entry shows the shortest distance, in miles, between two corresponding towns.

	W	R	T
Woebegone (W)	–	60	100
Rivendell (R)	60	–	80
Troy (T)	100	80	–

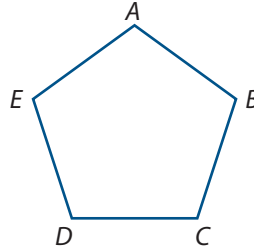
- a. Draw a vertex-edge graph that represents the information in the matrix.



- b. Use a compass and ruler to draw a scale diagram showing the distances between the towns. Assume straight-line roads between the towns.
- c. State a question involving these three towns that is best answered using the scale diagram.
- d. State a question that could be answered using either the scale diagram or the graph.

**14** The figure below is a *pentagon*. It has 5 vertices and 5 edges. Think about the figure as a vertex-edge graph.

- a. Write the adjacency matrix for this graph.
- b. Modify a copy of this graph by adding all the *diagonals*. (A diagonal is a line segment connecting 2 vertices that are not adjacent.) Write the adjacency matrix for this modified graph.
- c. Write a description of the adjacency matrix for a graph in the shape of a *polygon* with  $n$  sides. (A polygon with  $n$  sides is similar to the pentagon shown here, but there are  $n$  vertices and  $n$  edges.) How would you modify the description of the adjacency matrix if the graph consisted of the polygon *and* its diagonals?

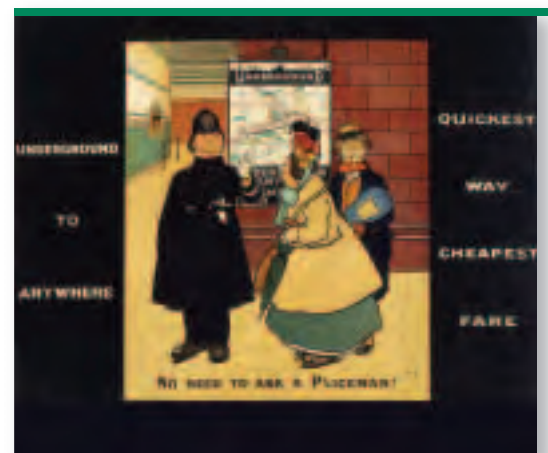


## Reflections

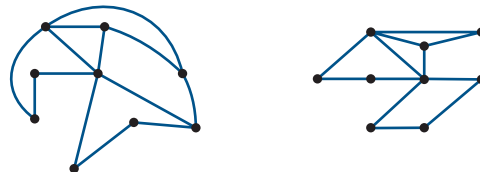
**15** Recall the original 1933 London Underground map on page 238. This is an example of a geometric diagram with vertices and edges for which precise size and shape is not crucial. The cartoon shown here accompanied the introduction of the map. The guide for the London Transport Museum describes the map as follows.

“Producing a map showing the different lines of the Underground system was particularly complicated. At first the Underground lines were shown geographically. A draughtsman, Harry Beck, devised his diagram in 1931. It uses only vertical, horizontal, or  $45^\circ$  diagonals and bears no relation to the real geography of London. At first the publicity department rejected it as too radical, finally publishing it in 1933 to instant enthusiasm from passengers. It was so simple and easy to use that the design is still used today and has been adapted by other cities around the world. It is a design classic.” (London Transport Museum Guide)

- a. Why do you think the publicity department thought the map was “too radical”?
- b. Why do you think the map was received with “instant enthusiasm” by the passengers?



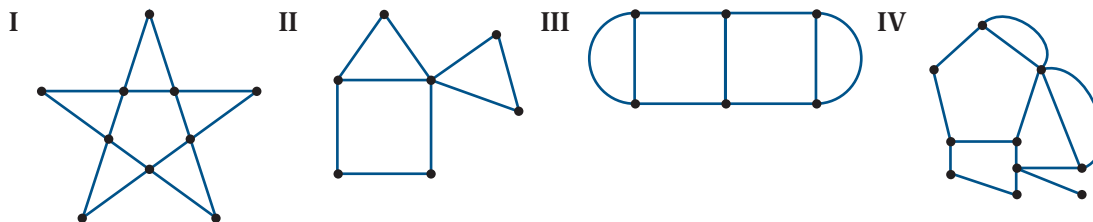
- 16** Two students have each drawn a graph that represents the combined routes of several school buses, as shown below. The vertices represent bus stops. Explain why the two graphs represent the same information. Label corresponding vertices with the same letter. If you have access to vertex-edge graph software, draw one of the graphs, then drag the vertices until it looks like the other graph.



- 17** Think of a problem different from those in this lesson that could be modeled with a vertex-edge graph and solved by using Euler circuits. Write a description of the problem and the solution.
- 18** Decide whether you agree with the following statement, and then write an argument to support your position: If a graph has an Euler circuit that begins and ends at a particular vertex, then it will have an Euler circuit that begins and ends at any vertex of the graph.
- 19** The algorithm for finding an Euler circuit in Problem 6 of Investigation 2 on page 245 is sometimes called the *onionskin algorithm*. Explain how this name describes what the algorithm does.
- 20** You might think that every matrix can be the adjacency matrix for some graph.
- a.** Try to draw graphs that have the following adjacency matrices.
- i.**  $\begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$       **ii.**  $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$       **iii.**  $\begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}$
- b.** Some matrices cannot be adjacency matrices for graphs. Write a description of the characteristics of a matrix that *could* be the adjacency matrix for a graph.

## Extensions

- 21** Graphs have interesting properties that can be discovered by collecting data and looking for patterns.



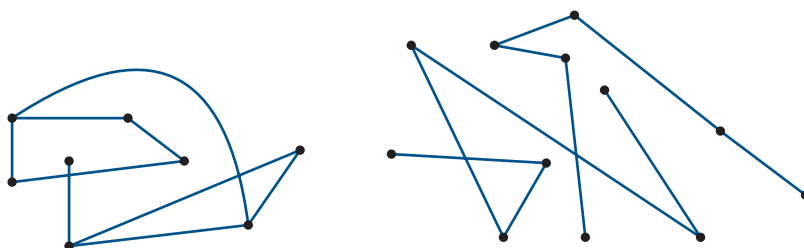
- a. Complete a table like the one that follows using the graphs above.

Graph	Sum of the Degrees of All Vertices	Number of Vertices of Odd Degree
I	30	
II		2
III		
IV		

- b. Write down any patterns you see in the table.
- c. See if the patterns continue when you collect more data. That is, draw a few more graphs, enter the information into the table, and check to see if the patterns you described in Part b are still valid. You might use vertex-edge graph software to help you generate graphs and information quickly.
- d. Explain why the sum of the degrees of all the vertices in *any* graph is an even number.
- e. Explain why *every* graph has an even number of vertices with odd degree.

- 22** Most of the graphs you have studied in this lesson have the key property of being *connected*. That is, they are all in one piece. Another way of thinking about connected graphs is that in a connected graph, there is a path from any vertex to any other vertex. Sometimes the way a graph is presented makes it difficult to tell whether or not it is connected.

- a. Determine if the following graphs are connected.



- b. Describe a systematic method you could use to check to see if a given graph is connected.

- 23** Decide whether each of the following statements is true (always true) or false (sometimes false). If a statement is true, explain as precisely as you can why it is true. If a statement is false, draw a *counterexample* that illustrates why it is false.
- Every vertex of a graph with an Euler circuit has degree greater than 1.
  - If every vertex of a graph has the same degree, the graph has an Euler circuit.

- 24** In this lesson, you discovered and used an important result about Euler circuits. Now think more carefully about that result.

- Explain as precisely as you can why this statement is true.

*If a graph has an Euler circuit, then  
all of its vertices have even degree.*

- Explain why or why not the following statement is true.

*If all the vertices of a connected graph have  
even degree, then the graph has an Euler circuit.*

- 25** Dominoes are rectangular tiles used to play a game. Each tile is divided into 2 squares with a number of dots in each square, as in the figure below.

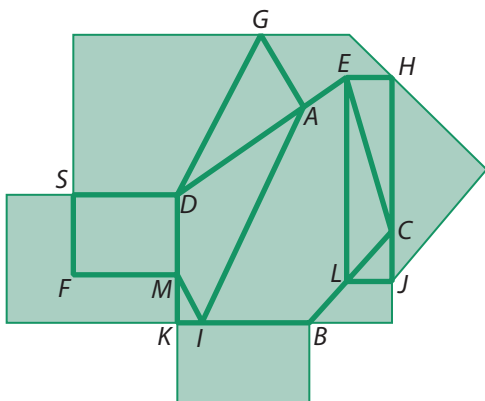


The standard set of dominoes has from 0 to 6 dots in each square. A deluxe set of dominoes has from 0 to 9 dots in each square. In each set, there is exactly one tile representing each possible number-pair combination. To play the game of dominoes, you take turns trying to place dominoes end-to-end by matching the number of dots. For example, for the three dominoes pictured below, the 3-5 domino can be placed next to the 1-3 domino, but the 0-2 domino cannot be placed next to either of the other dominoes.



Is it possible to form a ring of all the dominoes in a standard set placed end-to-end according to the rule above? How about for a deluxe set of dominoes? Explain your answers by reasoning about Euler circuits.

- 26** Euler circuits are also useful in manufacturing processes where a piece of metal is cut with a mechanical torch. To reduce the number of times the torch is turned on and off, it is desirable to make the cut continuous. For additional efficiency, the torch should not pass along an edge that has already been cut.





The metal piece must be clamped in air so that the torch does not burn the surface of the workbench. This leads to another condition; namely, any piece that falls off should not require additional cutting. Otherwise, it would have to be picked up and reclamped, a time-consuming process. Find a way to make all the cuts indicated on the pictured piece of metal, so that you begin and end at point S and the above conditions are satisfied.

- 27** RNA (ribonucleic acid) is a messenger molecule associated with DNA (deoxyribonucleic acid). RNA molecules consist of a chain of bases. Each base is one of 4 chemicals: U (uracil), C (cytosine), A (adenine), and G (guanine). It is difficult to observe exactly what an entire RNA chain looks like, but it is sometimes possible to observe fragments of a chain by breaking up the chain with certain enzymes. Armed with knowledge about the fragments, you can sometimes determine the makeup of the entire chain. One type of enzyme that breaks up an RNA chain is a “G-enzyme.” The G-enzyme will break an RNA chain after each G link. For example, consider the following chain.

AUUGCGAUC

A G-enzyme will break up this chain into the following fragments.

AUUG CG AUC

Unfortunately, the fragments of a broken-up chain are usually mixed up and in the wrong order. In this task, you will figure out how to reconstruct the chain when given some mixed-up fragments.

- a.** Suppose a different RNA chain is broken by a G-enzyme into the following fragments (although not necessarily in this order).

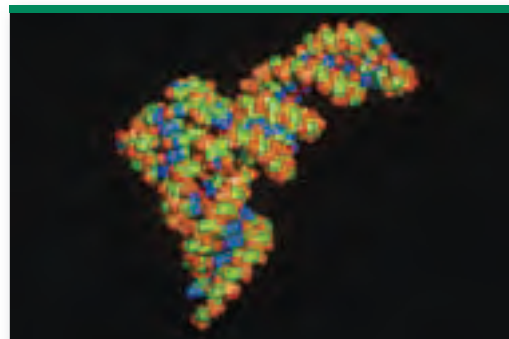
AUG AAC CG AG

Explain why the AAC fragment must be the end of the chain.

- b.** There is another enzyme, called a U-C enzyme, that breaks an RNA chain after each U or C link. For the unknown RNA chain in Part a, the U-C enzyme breaks the chain into the following fragments.

GC GAAC AGAU

As the final step in this chain-breaking process, the fragments are now further broken up using the other enzyme. That is, the fragments formed by the G-enzyme are now broken again, if possible, using the U-C enzyme, and vice versa. The resulting fragments from this process are shown in the table at the right. Each row of the table shows the break-up of each of the 7 fragments above. Complete the table by finding the rest of the final split fragments.



Computer-rendered molecular model of t-RNA

Original Fragment	Final Split Fragments
AUG	AU G
AAC	Not possible to split
CG	
AG	
GC	
GAAC	
AGAU	AG AU

- c. Mathematicians and biologists have discovered an amazing technique using Euler paths to reconstruct the unknown RNA chain. Carry out this technique, as follows:

**Step 1:** Draw vertices for each of the different final split fragments.

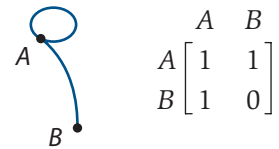
**Step 2:** Draw a **directed edge** (an arrow) from one vertex to another if the two split fragments are part of the same original fragment. The arrow should indicate how the two split fragments are recombined to get the original fragment.

**Step 3:** You now have a **directed graph**, that is, a graph where the edges have a direction. Find an Euler path through this graph (the start and end are not the same). Keep in mind that as you trace an Euler path, you must move in the direction shown by the directed edges.

**Step 4:** Put the fragments together as you traverse the Euler path. This will give you the original RNA chain.

28

A **loop** is an edge connecting a vertex to itself. When constructing an adjacency matrix for a graph with loops, a 1 is placed in the position in the matrix that corresponds with an edge joining a vertex to itself. An example of such a graph and its adjacency matrix is shown at the right.

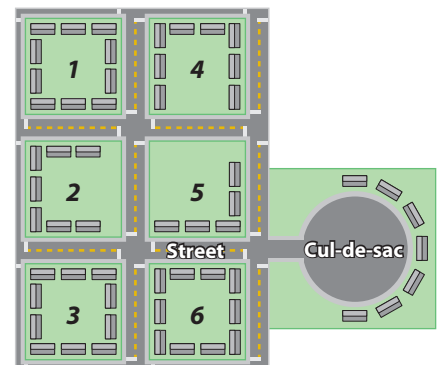


- a. Recall that the degree of a vertex is the number of edges touching the vertex, except that a loop counts for 2 edge touchings. What is the degree of vertex A?
- b. What is the row sum of the first row of the adjacency matrix above? In Investigation 3 of this lesson, you found a connection between row sums of an adjacency matrix and the degree of the corresponding vertex. Does this connection still hold for graphs with loops like the one above?

29

Some housing developments have houses built on a street that is a “cul-de-sac” so that traffic passing the houses is minimized.

- a. Suppose a cul-de-sac is located at the end of the street between blocks 5 and 6 as shown here. Draw a vertex-edge graph that represents this housing development.
- b. Find an optimum path for delivering papers to houses in this development.

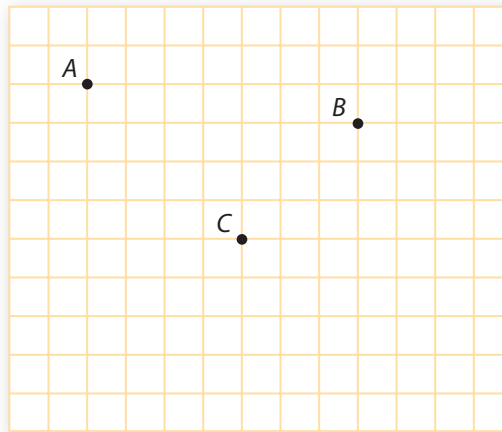


- c. You know from this lesson that the degree of a vertex is the number of edges that touch it, except that loops count as two edge touchings. Find the degree of each vertex in your graph.
- d. Repeat Parts a, b, and c with a second cul-de-sac constructed at the end of blocks 1 and 4.
- e. How does adding a cul-de-sac affect the graph? How does adding a cul-de-sac affect the optimum path for delivering papers to houses in the development?
- f. Does the condition about degrees of vertices for graphs with Euler circuits still hold for graphs with loops?

## Review

- 30** Although distance and position are not crucial features of vertex-edge graphs, these features are important in many geometric settings. Consider points in the grid below.

- a. What is the distance between points *C* and *B*? Between points *C* and *A*? Explain how to find those distances using the Pythagorean Theorem.
- b. Starting at point *C*, move 3 units to the right and 4 units down. Mark the point at this location. How far is this point from point *C*?
- c. Find a point that is exactly 10 units from point *A*, but is not directly above or below or directly to the right or left. What geometric shape is formed by all the points that are exactly 10 units from point *A*?



- 31** Place the following quantities in increasing order without using your calculator. Explain your method.

$$5\%, \frac{1}{10}, 0.5, \frac{1}{9}, \frac{4}{9}, 49\%$$

- 32** Study the relations represented in the following tables. If a relation is linear, find the slope of the graph representing the relationship.

**a.**

$x$	4	6	8	10	12
$y$	1	0	-1	-2	-3

**b.**

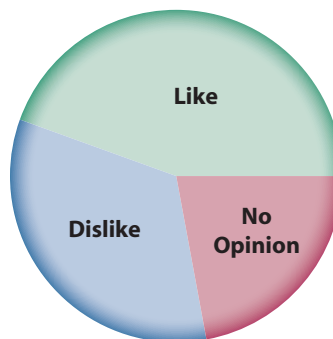
$x$	0	1	3	8
$y$	2	4	6	8

**c.**

$x$	0	0.1	0.2	0.3
$y$	3	2.95	2.9	2.85



- 33** In a recent poll, 630 students were asked if they like Chinese food. The circle graph below shows the results of the poll. Determine as precisely as possible how many people gave each response.



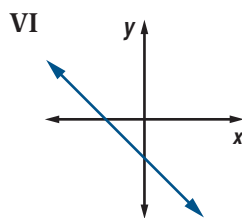
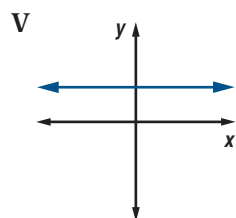
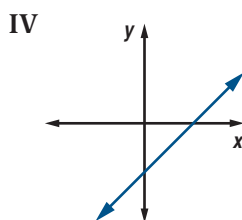
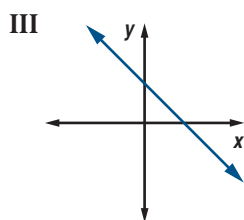
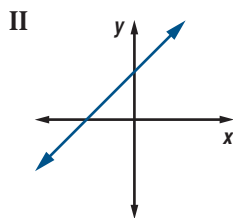
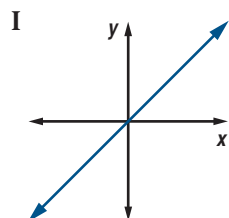
- 34** Complete a table like the one below showing some possible lengths, widths, and perimeters for a rectangle with an area of 24 square units.

$L$	1	2	3	4	5	6	7	8	9	10	11	12	24
$W$	24	12											
$P$	50	28											

- a.** Describe the pattern of change in  $W$  as  $L$  changes.  
**b.** Describe the pattern of change in  $P$  as  $L$  changes.  
**c.** Find formulas to represent  $W$  as a function of  $L$ , and  $P$  as a function of  $L$ .  
**d.** Are either of the patterns of change linear?
- 35** Solve each equation or inequality.
- a.**  $5x - 6 = 20$   
**b.**  $4.85 = 1.25x + 6.1$   
**c.**  $80 - \frac{3}{4}x < 20$   
**d.**  $75 \leq 15x + 100$

- 36** Without using your calculator, match each equation with a possible graph of the equation.

- a.  $y = x - 5$
- b.  $y = -x + 5$
- c.  $y = x$
- d.  $y = -x - 5$
- e.  $y = x + 5$
- f.  $y = 5$



- 37** Determine if the expressions in each pair are equivalent.

- a.  $6.02 \times 10^{21}$  and  $602 \times 10^{19}$
- b.  $980 \times 10^{10}$  and  $9.8 \times 10^8$
- c.  $0.034 \times 10^{12}$  and  $340 \times 10^8$

- 38** Draw sketches of a cone and a cylinder.



# LESSON 2

## *Vertex Coloring: Avoiding Conflict*

**I**n the last lesson, you learned about vertex-edge graphs. You used these graphs to model and solve problems related to paths and circuits. They can also be used to solve many other types of problems. In this lesson, you will investigate how graphs can be used to avoid possible conflict among a finite number of objects, people, or other things.

To begin, consider the problem of assigning radio frequencies to stations serving the same region. In cities and towns, you can listen to many different radio stations. Each radio station has its own transmitter that broadcasts on a particular channel, or frequency. The Federal Communications Commission (FCC) assigns the frequencies to the radio stations. The frequencies are assigned so that no two stations interfere with each other. Otherwise, you might tune into “Rock 101.7” and get Mozart instead!



## Think About This Situation

Suppose 7 new radio stations have applied for permits to start broadcasting in the same region of the country. Some stations may interfere with each other, others may not.

- What are some factors that may determine whether or not stations interfere with each other?
- How does this situation involve “conflict”?
- How do you think the FCC should assign frequencies to the 7 stations?
- Why do you think the FCC might like to assign the fewest possible number of new frequencies for the 7 stations?

In this lesson, you will learn how to use vertex-edge graphs to solve problems about avoiding conflicts, such as assigning noninterfering radio frequencies, using a technique called *vertex coloring*.

### Investigation 1 Building a Model

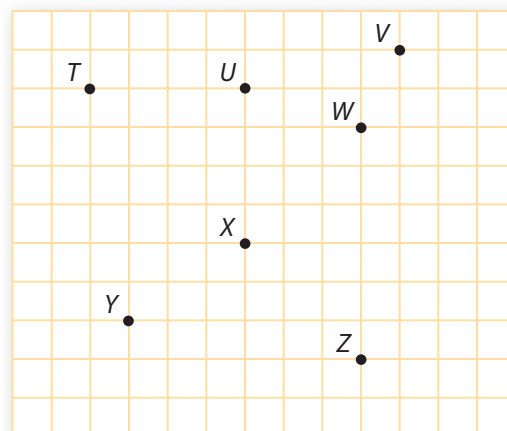
Suppose the 7 new radio stations that have applied for broadcast permits are located as shown on the grid below. A side of each small square on the grid represents 100 miles. The FCC wants to assign a frequency to each station so that no 2 stations interfere with each other. The FCC also wants to assign the fewest possible number of new frequencies. Suppose that because of geographic conditions and the strength of each station’s transmitter, the FCC determines that stations within 500 miles of each other will not interfere with each other.

Your work on the problems of this investigation will help you answer the question:

*How can vertex-edge graphs be used to assign frequencies to these 7 radio stations so that as few frequencies as possible are used and none of the stations interfere with each other?*

- For a small problem like this, you could solve it by trial and error. However, a more systematic method is needed for more complicated situations. Working on your own, begin modeling this problem with a graph. Remember, to model a problem with a graph, you must first decide what the vertices and edges represent.

a. What should the vertices represent?



Scale:  $\text{---} = 100$  miles

- b. How will you decide whether or not to connect 2 vertices with an edge? Complete this statement:

*Two vertices are connected by an edge if ...*

- c. Now that you have specified the vertices and edges, draw a graph for this problem.

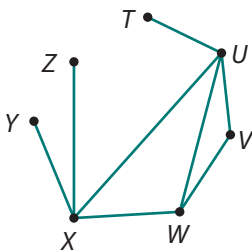
- 2 Compare your graph with those of your classmates.

- a. Did everyone define the vertices and edges in the same way? Discuss any differences.
- b. For a given situation, suppose two people define the vertices and edges in two different ways. Is it possible that both ways accurately represent the situation? Explain your reasoning.
- c. For a given situation, suppose two people define the vertices and edges in the same way. Is it possible that their graphs have different shapes but both are correct? Explain your reasoning.

- 3 A common choice for the vertices is to let them represent the radio stations. Edges might be thought of in two ways, as described in Parts a and b below.

- a. You might connect 2 vertices by an edge whenever the stations they represent are 500 miles or *less* apart. Did you represent the situation this way? If not, draw a graph where two vertices are connected by an edge whenever the stations they represent are 500 miles or *less* apart.
- b. You might connect 2 vertices by an edge whenever the stations they represent are *more* than 500 miles apart. Did you represent the situation this way? If not, draw a graph where 2 vertices are connected by an edge whenever the stations they represent are *more* than 500 miles apart.
- c. Compare the graphs from Parts a and b.
- Are both graphs accurate ways of representing the situation?
  - Which graph do you think will be more useful and easier to use as a mathematical model for this situation? Why?

- 4 For the rest of this investigation, use the graph where edges connect vertices that are 500 miles or less apart. Make sure you have a neat copy of this graph.



- a. Are vertices (stations) X and W connected by an edge? Are they 500 miles or less apart? Will their broadcasts interfere with each other?
- b. Are vertices (stations) Y and Z connected by an edge? Will their broadcasts interfere with each other?
- c. Compare your graph to the graph at the left.
- Explain why this graph also accurately represents the radio-station problem.

- ii. Describe what it means for two graphs to be “the same” even if their appearances are different.
- iii. If you have access to vertex-edge graph software, draw this graph and then drag its vertices so it looks like your graph in Problem 3, Part a.

**5** Remember that the problem is to assign frequencies so that there will be no interference between radio stations. So far, your graph models this problem as follows. Vertices represent the radio stations. Two vertices are connected by an edge if the corresponding radio stations are within 500 miles of each other. Here’s the last step in building the graph model—represent the frequencies as *colors*. So now, assigning frequencies to radio stations means to assign colors to the vertices.

Examine the statements in the following partially completed table. Translate each statement about stations and frequencies into a statement about vertices and colors. (The first one is already done for you.)

Statements about Stations and Frequencies	Statements about Vertices and Colors
Two stations have different frequencies.	Two vertices have different colors.
Find a way to assign frequencies so that stations within 500 miles of each other get different frequencies.	
Use the fewest number of frequencies.	

- 6** Now use as few colors as possible to **color the graph** for the radio-station problem. That is, assign a color to each vertex so that any 2 vertices that are connected by an edge have different colors. You can use colored pencils or just the names of some colors to do the coloring. Color or write a color code next to each vertex. Try to use the smallest number of colors possible.
- 7** Compare your coloring with another student’s coloring.
- a. Do both colorings satisfy the condition that vertices connected by an edge must have different colors?
  - b. Do both colorings use the same number of colors to color the vertices of the graph? Reach agreement about the minimum number of colors needed.
  - c. Explain, in writing, why the graph cannot be colored with fewer colors.
  - d. For 2 particular vertices, suppose one student colors both vertices red while another student colors 1 vertex red and the other blue. Is it possible that both colorings are acceptable? Explain your reasoning.
  - e. Describe the connection between graph coloring and assigning frequencies to radio stations.



## Summarize the Mathematics

Some problems can be solved by coloring the vertices of an appropriate graph.

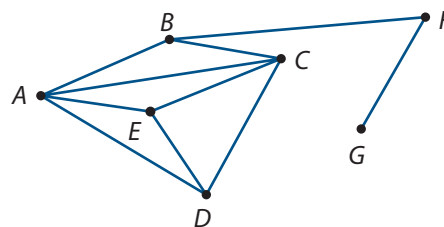
- a What do the vertices, edges, and colors represent in the graph that you used to solve the radio-station problem?
- b How did “coloring a graph” help solve the radio-station problem?
- c In what ways can two graphs differ and yet still both accurately represent a situation?

**Be prepared to share your ideas with the class.**

### ✓ Check Your Understanding

Consider the graph at the right.

- a. On a copy of the graph, color the vertices using as few colors as possible.
- b. If possible, find a second coloring of the graph in which some of the vertices colored the same in Part a are no longer colored the same. Again use as few colors as possible.



### Investigation 2

#### Scheduling, Mapmaking, and Algorithms

Now that you know how to color a graph, you can use graph coloring to solve many other types of problems. As you work on the problems in this investigation, look for answers to this question:

*What are the basic steps of modeling and solving conflict problems using vertex-edge graphs?*

**Scheduling Meetings** There are 6 clubs at King High School that want to meet once a week for one hour, right after school lets out. The problem is that several students belong to more than one of the clubs, so not all the clubs can meet on the same day. Also, the school wants to schedule as few days per week for after-school club meetings as possible. Below is the list of the clubs and the club members who also belong to more than one club.



## Clubs and Members

Club	Students Belonging to More Than One Club
Varsity Club	Christina, Shanda, Carlos
Math Club	Christina, Carlos, Wendy
French Club	Shanda
Drama Club	Carlos, Vikas, Wendy
Computer Club	Vikas, Shanda
Art Club	Shanda

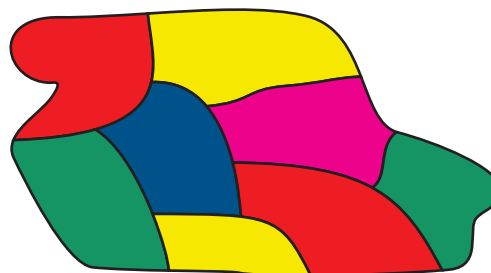


- 1** Consider the club-scheduling problem as a graph-coloring problem.
  - a.** Your goal is to assign a meeting day (Monday–Friday) to each club in such a way that no 2 clubs that share a member meet on the same day. Also, you want to use as few days as possible. Working on your own, decide what you think the vertices, edges, and colors should represent.
  - b.** Compare your representations with others. Decide as a group which representations are best. Complete these three statements.  
*The vertices represent ... .*  
*Two vertices are connected by an edge if ... .*  
*The colors represent ... .*
  - c.** Draw a graph that models the problem.
  - d.** Color the club-scheduling graph using as few colors as possible.
- 2** Use your graph coloring in Problem 1 to answer these questions.
  - a.** Is it possible for every club to meet once per week?
  - b.** What is the fewest number of days needed to schedule all the club meetings?
  - c.** On what day should each club meeting be scheduled?
  - d.** Explain how your coloring of the graph helps you answer each of the questions above.

**Coloring Maps** Another class of problems for which graph coloring is useful involves coloring maps. You may have noticed in your geography or social-studies course that maps are always colored so that neighboring countries do not have the same color. This is done so that the countries are easily distinguished and don't blend into each other. In the following problems, you will explore the number of different colors necessary to color *any* map in such a way that no 2 countries that share a border have the same color. This is a problem that mathematicians worked on for many years, resulting in a lot of new and useful mathematics. For this problem, *countries* are assumed to be regions that are contiguous (not broken up into separate parts), and *border* means a common boundary of some length (touching at points doesn't matter).



- 3** Shown here is an uncolored map of a portion of southern Africa.
- Using a copy of this map, color the map so that no 2 countries that share a border have the same color. (For this problem, you may assume that Botswana and Zambia intersect only at a point, though in fact they share a border about 10 miles long.)
  - How many colors did you use? Try to color the map with fewer colors.
  - Compare your map coloring with that of other classmates.
    - Are the colorings different?
    - Are the colorings legitimate; that is, do neighboring countries have different colors? If a coloring is not legitimate, fix it.
  - What is the fewest number of colors needed to color this map?
- 4** In Problem 3, you found the fewest number of colors needed to color the Africa map. Now think about the fewest number of colors needed to color *any* map.
- Do you think you can color *any* map with at most 5 different colors? Can the map of Africa be colored with 5 colors?
  - The map below has been colored with 5 colors. Is it possible to color the map with fewer than 5 colors? If so, make a copy of the map and color it with as few colors as possible.



- What do you think is the *fewest* number of colors needed to color *any* map? Make a conjecture.
  - Compare your conjecture to the conjectures of your classmates. Briefly discuss any differences. Revise your conjecture if you think you should. Test your conjecture as follows, in part ii.
  - Over the next few days, test your conjecture on other maps outside of class. Revise your conjecture as necessary. Compare and discuss again with your classmates and teacher.

Maps can be colored by working directly with the maps, as you have been doing. But it is also possible to turn a map-coloring problem into a graph-coloring problem. This can be helpful since it allows you to use all the properties and techniques for graphs to help you understand and solve map-coloring problems.

**5** To find a graph that models a map-coloring problem, first think about what you did with the radio-station and club-scheduling problems. In both of those problems, the edges were used to indicate some kind of *conflict* between the vertices. The vertices in conflict were connected by an edge and colored different colors. A crucial step in building a graph-coloring model is to decide what the conflict is. Once you know the conflict, you can figure out what the vertices, edges, and colors should represent.

- a. What was the conflict in the club-scheduling problem? What was the conflict in the radio-station problem?
- b. Make and complete a table like the one below.

### Modeling Conflicts

Problem	Conflict if:	Vertices	Connect with an Edge if:	Colors
Radio-station problem	2 radio stations are 500 miles or less apart	radio stations		frequencies
Club-scheduling problem	two clubs _____			
Map-coloring problem	two countries _____			

- 6** Consider the map of a portion of southern Africa in Problem 3.
- a. Use the information in the table above to create a vertex-edge graph that represents the map.
  - b. Color the vertices of the graph. Remember that coloring always means that vertices connected by an edge must have different colors. Also, as usual, use as few colors as possible.
  - c. Compare your coloring with those of other classmates.
    - i. Are all the colorings legitimate?
    - ii. Reach agreement on the fewest number of colors needed to color the graph.
    - iii. Is the minimum number of colors for this *graph*-coloring problem the same as the minimum number of colors for the *map*-coloring problem in Part d of Problem 3? Explain.

**Algorithms** You have now used vertex coloring to solve several problems. In each problem, you used some method to color the vertices of a graph using as few colors as possible. Recall that a systematic step-by-step method is called an *algorithm*. Finding good graph-coloring algorithms is an active area of mathematical research with many applications. It has proven quite difficult to find an algorithm that colors the vertices of any graph using as few colors as possible. You often can figure out how to do this for a given small graph, as you have done in this lesson. However, no one knows an efficient algorithm that will color *any* graph with the *fewest* number of colors! This is a famous unsolved problem in mathematics. Think about methods you have used to color a graph.

**7** Describe some strategies or algorithms you have used to color the vertices of a graph using the fewest number of colors. Compare and discuss algorithms with your classmates.

**8** One commonly used algorithm is sometimes called the **Welsh-Powell algorithm**. Here's how it works:

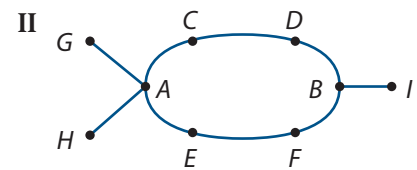
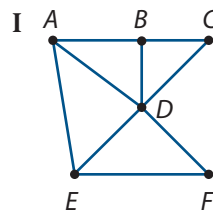
**Step 1:** Begin by making a list of all the vertices starting with the ones of highest degree and ending with those of lowest degree.

**Step 2:** Color the first uncolored vertex on your list with an unused color.

**Step 3:** Go down the list coloring as many uncolored vertices with the current color as you can, following the rule that vertices connected by an edge must be different colors.

**Step 4:** If all the vertices are now colored, you're done. If not, go back to Step 2.

**a.** Follow the Welsh-Powell algorithm, step by step, to color the two graphs below.



**b.** Does the Welsh-Powell algorithm always yield a coloring that uses the fewest number of colors possible? Explain your reasoning.

**c.** Use the Welsh-Powell algorithm to color each graph below and compare your coloring with your previous results.

**i.** radio-station graph (Investigation 1, Problem 6)

**ii.** club-scheduling graph (Investigation 2, Problem 1, Parts c and d)

## Summarize the Mathematics

In this lesson, you used graphs to avoid conflict in three seemingly different problems.

- a** Explain how each of the three main problems you solved in this lesson—assigning radio frequencies, club scheduling, and map coloring—involved “conflict among a finite number of objects.”
- b** Describe the basic steps of modeling and solving a conflict problem with a graph.
- c** The **chromatic number** of a graph is the fewest number of colors needed to color all its vertices so that 2 vertices connected by an edge have different colors. What is the chromatic number of the graphs for each of the three problems in this lesson (radio stations, club meetings, and map coloring)? How is this number related to the solution for each problem?

*Be prepared to share your responses with the entire class.*

### Check Your Understanding

Hospitals must have comprehensive and up-to-date evacuation plans in case of an emergency. A combination of buses and ambulances can be used to evacuate most patients. Of particular concern are patients under quarantine in the contagious disease wards. These patients cannot ride in buses with nonquarantine patients. However, some quarantine patients can be transported together. The records of who can be bused together and who cannot are updated daily.

Suppose that on a given day there are 6 patients in the contagious disease wards. The patients are identified by letters. Here is the list of who cannot ride with whom:

- |   |                                   |
|---|-----------------------------------|
| $A$ cannot ride with $B$ , $C$ , or $D$ . | $D$ cannot ride with $A$ or $C$ . |
| $B$ cannot ride with $A$ , $C$ , or $E$ . | $E$ cannot ride with $F$ or $B$ . |
| $C$ cannot ride with $A$ , $B$ , or $D$ . | $F$ cannot ride with $E$ .        |

The problem is to determine how many vehicles are needed to evacuate these 6 patients. Use graph coloring to solve this problem. Describe the conflict, and state what the vertices, edges, and colors represent.

# On Your Own

## Applications

- 1 A nursery and garden center plants a certain number of “mix-and-match” flower beds. Each bed contains several different varieties and colors. This allows customers to see possible arrangements of flowers that they might plant.



However, the beds are planted so that no bed contains two colors of the same variety. For example, no bed contains both red roses and coral roses. Also, no bed contains two varieties of the same color. For example, no bed contains both yellow tulips and yellow marigolds. This is done so that the customer can distinguish among and appreciate the different colors and varieties. A list of the varieties and colors that will be planted follows.

**Flower Beds**

Varieties	Colors
Roses	Red, Coral, White
Tulips	Yellow, Purple, Red
Marigolds	Yellow, Orange

The nursery wants to plant as few mix-and-match beds as possible. In this task, you will determine the minimum number of mix-and-match flower beds.

- The varieties and colors listed above yield 8 different types of flowers, such as red roses, red tulips, and yellow tulips. List all the other types of flowers that are possible.
- It is the types of flowers from Part a that will be planted in the mix-and-match beds. The problem is to figure out the minimum number of beds needed to plant these types of flowers so that no bed contains flowers that are the same variety or the same color. First, you need to build a graph-coloring model.
  - What should the vertices represent?
  - What should the edges represent? Why?
  - What should the colors of the graph represent?
- Draw the graph model and color it with as few colors as possible.



- d. What is the minimum number of mix-and-match beds needed?
- e. Use your graph coloring to recommend to the nursery which types of flowers should go in each of the mix-and-match beds.
- f. When using a graph-coloring model, you connect vertices by an edge whenever there is some kind of conflict between the vertices. What was the conflict in this task?

- 2 A local zoo wants to take visitors on animal-feeding tours. They propose the following tours.

**Tour 1** Visit lions, elephants, buffaloes

**Tour 2** Visit monkeys, hippos, deer

**Tour 3** Visit elephants, zebras, giraffes

**Tour 4** Visit hippos, reptiles, bears

**Tour 5** Visit kangaroos, monkeys, seals

The animals are fed only once a day. Also, there is only room for 1 tour group at a time at any 1 site. What is the fewest number of days needed to schedule all 5 tours? Explain your answer in terms of graph coloring.

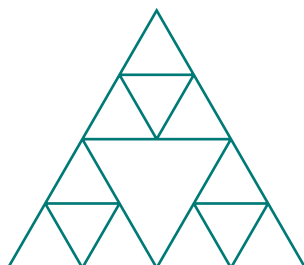


- 3 You often can color small maps directly from the map, without translating to a graph model. However, using a graph model is essential when the maps are more complicated. The map of South America shown here can be colored either directly or by using a graph-coloring model.



- a. Color a copy of the map of South America directly. Use as few colors as possible and make sure that no two bordering countries have the same color.
- b. Represent the map as a graph. Then color the vertices of the graph with as few colors as possible.
- c. Did you use the same number of colors in Parts a and b?

- 4 The following figure is part of what is called a *Sierpinski Triangle*. (The complete figure is actually drawn by an infinite process described in Extensions Task 18 on page 283.)

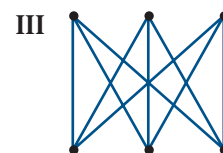
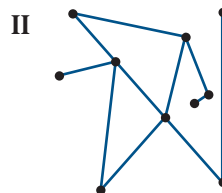
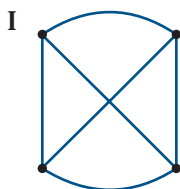
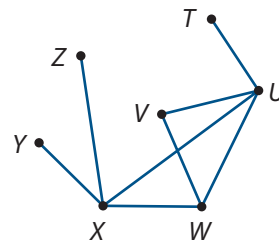


- a. Think of this figure as a map in which each triangle *not* containing another triangle is a country. Make and color a copy of the map with as few colors as possible.
- b. Construct a graph that represents this map. Color the vertices of the graph with as few colors as possible. Compare the number of colors used with that in Part a.
- c. Think of this figure as a map as Sierpinski did: the triangles with points upwards are countries, and the triangles with points downwards are bodies of water, “Sierpinski oceans,” that separate the countries. Using this interpretation of countries, color a copy of the map with as few colors as possible. (Leave the Sierpinski oceans uncolored.)
- d. Construct a graph for this second map. Color this graph with as few colors as possible. Did you use the same number of colors as in Part c?

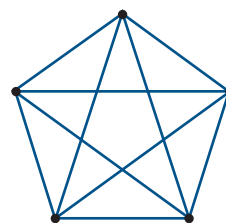
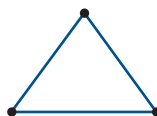
## Connections

- 5 Shown here is a student’s proposal for a graph that models the radio-station problem from page 267.

- a. Is this a legitimate model for the radio-station problem? Explain your reasoning.
- b. In this graph model, some edges intersect at places that are not vertices. Can the graph be redrawn without edge-crossings? If so, do so.
- c. Graphs that *can* be drawn in the plane with edges intersecting only at the vertices are called **planar graphs**. Which of the graphs below are planar graphs? (Use available software to demonstrate).



- 6 This task explores some properties of *complete graphs*. A **complete graph** is a graph that has exactly one edge between every pair of vertices. Complete graphs with 3 and 5 vertices are shown below.



- a. Draw a complete graph with 4 vertices. Draw a complete graph with 6 vertices.

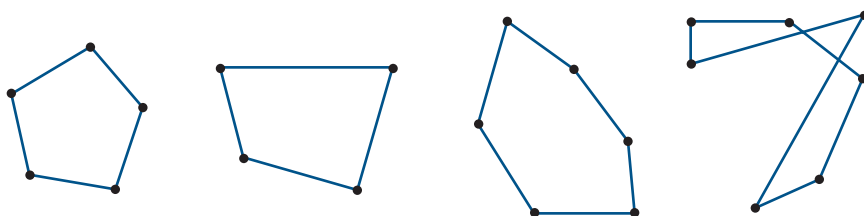
- b. Make a table that shows the number of edges for complete graphs with 3, 4, 5, and 6 vertices.
- c. Look for a pattern in your table. How many edges does a complete graph with 7 vertices have? A complete graph with  $n$  vertices?

**7** Refer to the definition of a complete graph given in Connections Task 6.

- a. What is the minimum number of colors needed to color the vertices of a complete graph with 3 vertices? A complete graph with 4 vertices? A complete graph with 5 vertices?
- b. Make a table showing the number of vertices and the corresponding minimum number of colors needed to color a complete graph with that many vertices. Enter your answers from Part a into the table. Find two more entries for the table.
- c. Describe any patterns you see in the table.
- d. What is the minimum number of colors needed to color a complete graph with 100 vertices? With  $n$  vertices?

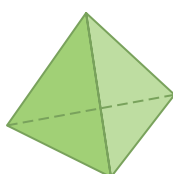
**8** A **cycle graph** is a graph consisting of a single *cycle* (a route that uses each edge and vertex exactly once and ends where it started).

- a. Color the vertices of each of the cycle graphs below using as few colors as possible.

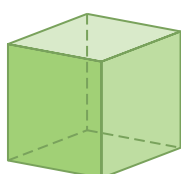


- b. Make a conjecture about the minimum number of colors needed to color cycle graphs. Write an argument supporting your conjecture. Test your conjecture by drawing and coloring some large cycle graphs using vertex-edge graph software if available.

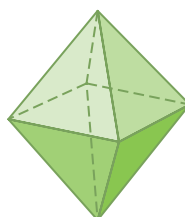
**9** Besides coloring graphs, it is also possible to color other geometric figures. The three-dimensional figures below are three of the five *regular polyhedra*. You will learn more about regular polyhedra in Unit 6 *Patterns in Shape*. For now, you just need to visualize these three objects.



**Tetrahedron**



**Hexahedron**



**Octahedron**

Complete Parts a, b, and c for each of the above polyhedra. Record your answers for each of these coloring schemes in a table like the one on the next page.

### Coloring Polyhedra

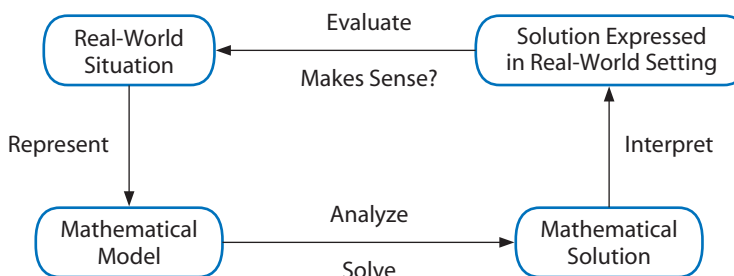
Regular Polyhedron	Minimum Number of Colors		
	for Vertices	for Edges	for Faces
Tetrahedron			
Hexahedron			
Octahedron			

- Color the vertices. Use the minimum number of colors. (Vertices connected by an edge must have different colors.)
- Color the edges. Use the minimum number of colors. (Edges that share a vertex must have different colors.)
- Color the faces. Use the minimum number of colors. (Faces that are adjacent must have different colors.)

## Reflections

- Throughout this course, and this unit in particular, you have been doing *mathematical modeling*. Below is a diagram that summarizes the process of mathematical modeling.

### Process of Mathematical Modeling



Choose one example of mathematical modeling from this lesson. Use the example to illustrate each part of the diagram.

- Think of a problem situation different from any in this lesson that could be solved by vertex coloring. Describe the problem and the solution.
- In this lesson, as well as in previous units, you have engaged in important kinds of mathematical thinking. From time to time, it is helpful to step back and reflect on the kinds of thinking that are broadly useful in doing mathematics. Look back over Lessons 1 and 2 and consider some of the mathematical thinking you have done. Describe an example where you did each of the following.
  - Search for and describe patterns
  - Formulate or find a mathematical model
  - Make and check conjectures
  - Describe and use algorithms
  - Use different representations of the same idea

## Extensions

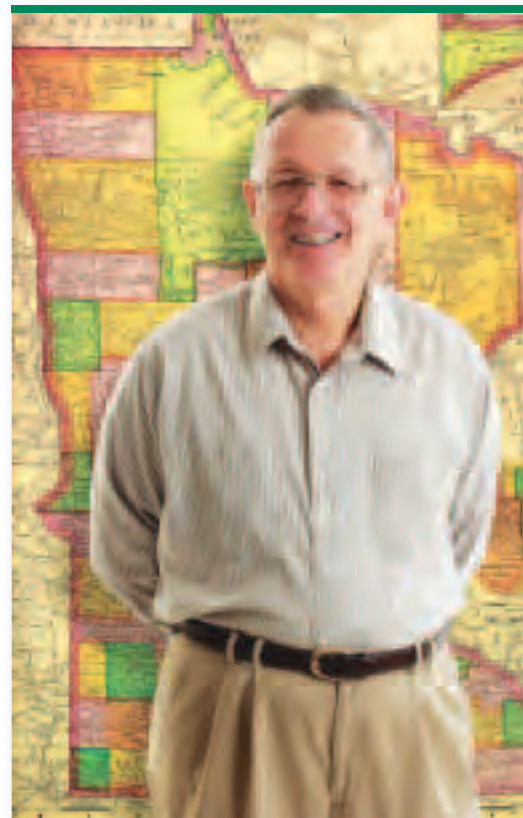
- 13** Search the Internet or a library for information on mathematicians who have worked on map coloring. Write a one-page report on one mathematician's contribution to the field.
- 14** In the nineteenth century, mathematicians made a conjecture about the minimum number of colors needed to color any map so that regions with a common boundary have different colors. This conjecture became one of the most famous unsolved problems in mathematics—until 1976 when the problem was solved. Based on your work in this lesson, how many colors do you think are needed to color *any* map? Only consider maps where the regions are connected. So, for example, do not consider a map that has a country that is split into two parts separated by another country.
- Try to draw a map that requires 3 colors and cannot be colored with fewer colors.
  - Try to draw a map that requires 4 colors and cannot be colored with fewer colors.
  - Try to draw a map that requires 5 colors and cannot be colored with fewer colors.
  - How many colors do you think are necessary to color any map? After you have worked on this problem for a while, search the Internet or a library for recent information on graph theory and map coloring. Find the answer and compare it to your answer. Write a brief report on your findings.
- 15** In Problem 7 on page 274, you described algorithms that you used to color a graph. You may have described one of the following two algorithms (adapted from the description in *Discrete Algorithmic Mathematics, 3rd Edition*, A K Peters, Ltd, 2004, page 294).

### Vertex-by-Vertex Algorithm

- Step 1:** Arbitrarily number all the vertices of the graph: Vertex 1, Vertex 2, Vertex 3, and so on. Also, number the colors: Color 1, Color 2, and so on.
- Step 2:** Color the first vertex on your list with the first color.
- Step 3:** Color the next vertex on your list with the lowest-numbered color not already used for an adjacent vertex.
- Step 4:** Continue vertex-by-vertex until all vertices are colored.

### One-Color-at-a-Time Algorithm

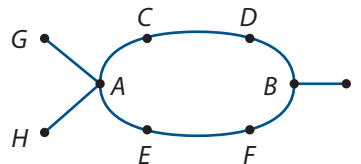
- Step 1:** Arbitrarily number all the vertices of the graph: Vertex 1, Vertex 2, Vertex 3, and so on.
- Step 2:** Color the lowest-numbered vertex with an unused color.
- Step 3:** Go down the list of vertices coloring as many uncolored vertices with the current color as you can, following the rule that adjacent vertices must be different colors.
- Step 4:** If all the vertices are now colored, you're done. If not, go back to Step 2.



Ken Appel and colleague Wolfgang Haken of the University of Illinois used 1,200 hours of computer time to help solve the map-coloring problem.



- a. Color the graph below (from Problem 8 on page 274) using each of the two algorithms above. For this problem, number the vertices in the same way for each algorithm.



- b. Compare the colorings from Part a.
- Make a conjecture about the colorings produced by these two algorithms for any graph.
  - Use vertex-edge graph software to test your conjecture with other graphs. For example, load some of the specific graph examples from the Sample Graphs menu. Then color using both algorithms, as found in the Algorithms menu. Each time you apply one of these algorithms, the software creates a random numbering of the vertices of the graph. What happens when the two algorithms are used with the same numbering of the vertices?
  - Explain why you think your conjecture is true.
- c. Compare the One-Color-at-a-Time algorithm to the Welsh-Powell algorithm in Problem 8 on page 274. Describe similarities and differences.
- d. Use vertex-edge graph software to help you further investigate the three algorithms you have considered in this task: the Vertex-by-Vertex algorithm, the One-Color-at-a-Time algorithm, and the Welsh-Powell algorithm. Explore the following questions. Write a brief report summarizing your findings. Include examples, counterexamples, and explanations as needed.
- What happens when you change the order in which you list the vertices? Do some types of orderings seem to generally be better than others?
  - Does one algorithm always yield a coloring with fewer colors than the others?
  - Will any of the algorithms always produce the chromatic number when applied to any graph?



16

In this lesson, coloring a graph has always meant coloring the *vertices* of the graph. It also can be useful to think about **coloring the edges** of a graph. For example, suppose there are 6 teams in a basketball tournament and each team plays every other team exactly once. Games involving different pairs of teams can be played during the same round, that is, at the same time. The problem is to figure out the fewest number of rounds that must be played. One way to solve this problem is to represent it as a graph and then color the *edges*.

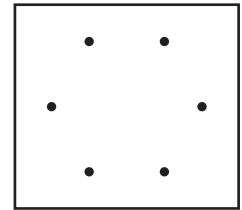
- a. Represent the teams as vertices. Connect 2 vertices with an edge if the 2 teams will play each other in the tournament. Draw the graph model.



- b. Color the edges of the graph so *edges that share a vertex have different colors*. Use as few colors as possible.
- c. Think about what the colors mean in terms of the tournament and the number of rounds that must be played. Use the edge coloring to answer these questions.
  - i. What is the fewest number of rounds needed for the tournament?
  - ii. Which teams play in which rounds?
- d. Describe another problem situation that could be solved by edge coloring.

**17** Here is an interesting game involving a type of edge coloring that you can play with a friend.

- Place 6 points on a sheet of paper to mark the vertices of a *regular hexagon*, as shown here.
  - Each player selects a color different from the other.
  - Take turns connecting 2 vertices with an edge. Each player should use his or her color when adding an edge.
  - The first player who is forced to form a triangle of his or her own color loses. (Only triangles with vertices among the 6 starting vertices count.)
- a. Play this game several times and then answer the questions below.
    - i. Is there always a winner?
    - ii. Which player has the better chance of winning? Explain.
  - b. Use the results of Part a to help you solve the following problem.  
*Of any 6 students who are in a room, must there be at least 3 mutual acquaintances or at least 3 mutual strangers?*



**18** The Sierpinski Triangle is a very interesting geometric figure. If you try to draw it, you will never finish. That's because it is defined by an infinitely repetitive set of instructions. Here are the instructions.

**Step 1:** Draw an equilateral triangle.

**Step 2:** Find the midpoint of each side.

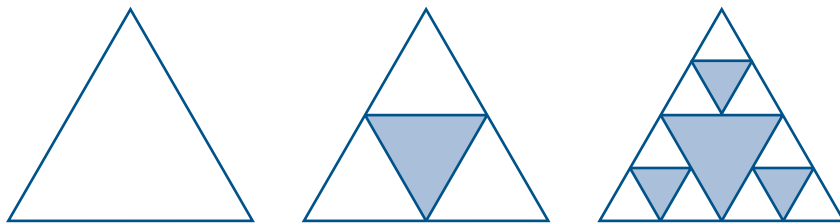
**Step 3:** Connect the midpoints. This will subdivide the triangle into 4 smaller triangles.

**Step 4:** Remove the center triangle. (Don't actually cut it out, just think about it as being removed. If you wish, you can shade it with a pencil to remind yourself it has been "removed.") Now there are 3 smaller triangles left.

**Step 5:** Repeat Steps 2–4 with each of the remaining triangles. Continue this process with successively smaller triangles. The first two passes through the instructions are illustrated at the top of the next page.



Sierpinski Triangle quilt  
made by Diana Venters.



If you continue this process, you never get finished with these instructions because there always will be smaller and smaller triangles to subdivide.

- a. On an enlarged copy of the third stage, draw the next stage of the process.
- b. Stretch your imagination and think of the Sierpinski Triangle as a map, where the countries are the triangles that don't get removed. What is the minimum number of colors needed to color the map?

## Review

- 19** Each table below represents a linear relation. Complete the table and find the equation of the line.

a.

<i>x</i>	0	1	2	3	4	5	6
<i>y</i>	0			7			

b.

<i>x</i>	0	1	2	3	4	5	6
<i>y</i>	4			7			

c.

<i>x</i>	0	1	2	3	4	5	6
<i>y</i>				7			7

d.

<i>x</i>	0	1	2	3	4	5	6
<i>y</i>				7		3	

- 20** Try to answer these questions without the use of a calculator. Think about how your answer for one part can help you determine an answer for another part.

- a. What percent is 80 of 800?
- b. What percent is 8 of 800?
- c. What percent is 0.8 of 800?
- d. What percent is 4 of 800?
- e. What percent is 1 of 800?
- f. What percent is 0.5 of 800?

- 21** Solve  $3(x - 1) = 7x + 5$  by any method. Explain your method.

- 22** Donna wants to buy a painting that regularly sells for a price of \$55 but is on sale for 20% off. If the sales tax is 7%, how much money will Donna need in order to buy the painting?

**23** Sketch a graph of each equation.

a.  $y = 3x + 4$

b.  $y = -\frac{2}{3}x + 6$

c.  $y = -2 + \frac{1}{4}x$

d.  $y = 3$

**24** Without using a calculator, find the value of each expression.

a.  $-5^2 + 10$

b.  $3^3 - 2^3$

c.  $(-2)^2 - 4(-5)$

d.  $6(4)^3$

e.  $8(0.5)^2$

f.  $\frac{3(2^3)}{6}$

**25** Calculate the area of each shape.

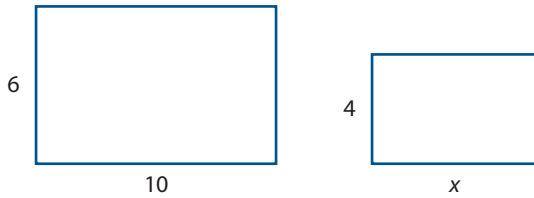
a. a right triangle that has a base of 7 inches and a height of 4 inches

b. a parallelogram with length 8 cm and height 5 cm

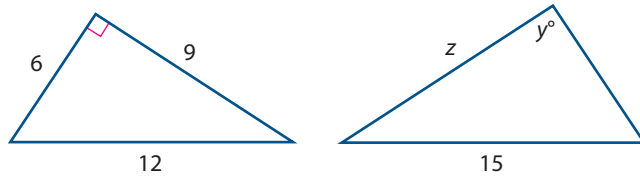
c. a square with perimeter 64 feet

**26** Assume that the polygons in each pair shown below are similar with corresponding sides and angles as suggested by the diagrams. Find the unknown side lengths and angle measurements  $x$ ,  $y$ ,  $z$ ,  $p$ ,  $w$ , and  $t$ .

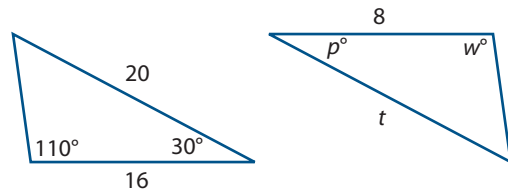
a.



b.



c.



# LESSON 3

## Looking Back

**I**n this unit, you have studied a type of geometric diagram consisting of vertices and edges called a *graph*, or sometimes *vertex-edge graph*. The essential characteristic of these graphs is the relationship among the vertices, as defined by how the edges connect the vertices. These vertex-edge graphs can be used as models to help understand and solve many interesting types of problems.

You have used Euler circuits and vertex coloring to find optimum circuits and to manage conflicts in a variety of settings. The tasks in this final lesson will help you review and organize your thinking about the use of vertex-edge graphs as mathematical models.

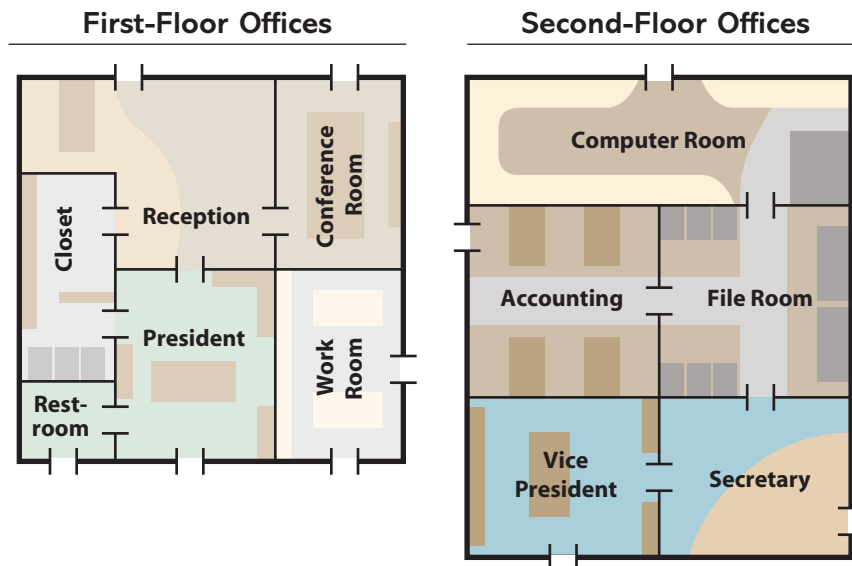
- 1** One city's Department of Sanitation organizes garbage collection by setting up precise garbage truck routes. Each route takes one day. Some sites that need garbage collection more often are on more than one route. However, if a site is on more than one route, the routes should not visit that site on the same day. Here is a list of routes and the sites on each route that are also on other routes.

Route 1: Site A, Site C  
Route 2: Site D, Site A, Site F  
Route 3: Site C, Site D, Site G  
Route 4: Site G  
Route 5: Site B, Site F  
Route 6: Site D  
Route 7: Site C, Site F, Site B

- a.** Can all 7 routes be scheduled in one week (Monday–Friday)? What is the fewest number of days needed to schedule all 7 routes?
- b.** Set up a schedule for the garbage truck routes, showing which routes run on which day of the week.

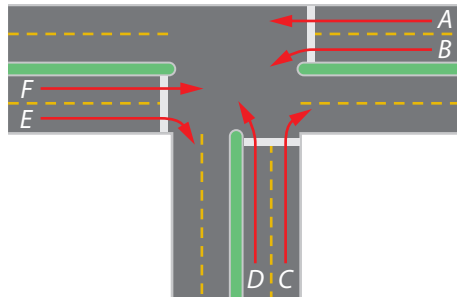


- 2 The security guard for an office building must check the building several times throughout the night. The diagrams below are the floor plans for office complexes on two floors of the building. An outer corridor surrounds each office complex. In order to check the electronic security system completely, the guard must pass through each door at least once.



- For each office complex, can the guard walk through each door exactly once, starting and ending in the outer corridor? If so, show the route the guard could take. If not, explain why not.
- If it is not possible to walk through each door exactly once starting and ending in the outer corridor, what is the fewest number of doors that need to be passed through more than once? Show a route the guard should take. Indicate the doors that are passed through more than once.
- Construct an adjacency matrix for the graph modeling the first-floor offices problem. Explain how to use the matrix to solve the problem.

- 3 Traffic lights are essential for controlling the flow of traffic on city streets, but nobody wants to wait at a light any longer than necessary. Consider the intersection diagrammed below. The arrows show the streams of traffic. There is a set of traffic lights in the center of the intersection.



- a. Can traffic streams  $B$  and  $D$  have a green light at the same time? How about  $B$  and  $C$ ? List all the traffic streams that conflict with  $B$ .
- b. Streams of traffic that have a green light at the same time are said to be on the same green-light cycle. What is the fewest number of green-light cycles necessary to safely accommodate all 6 streams of traffic?
- c. For each of the green-light cycles you found in Part b, list the streams of traffic that can be on that cycle.

## Summarize the Mathematics

In this unit, you have used vertex-edge graphs as mathematical models to help solve a variety of problems.

- a When constructing a mathematical model, you look for and mathematically represent the essential features of a problem situation. For each of the three tasks in this lesson, describe the essential features of the problem situation and how they are represented in the graph model you used. Be sure to describe what the vertices, edges, and colors (if needed) represent in each case.
- b Key mathematical topics in this unit are Euler circuits and vertex coloring.
  - i. What is an Euler circuit?
  - ii. How can you tell if a graph has an Euler circuit?
  - iii. Describe the types of problems that can be solved with Euler circuits.
  - iv. Describe what it means to “color the vertices of a graph.”
  - v. Describe the types of problems that can be solved by vertex coloring.

*Be prepared to share your descriptions and reasoning with the class.*

## Check Your Understanding

Write, in outline form, a summary of the important mathematical concepts and methods developed in this unit. Organize your summary so that it can be used as a quick reference in future units and courses.